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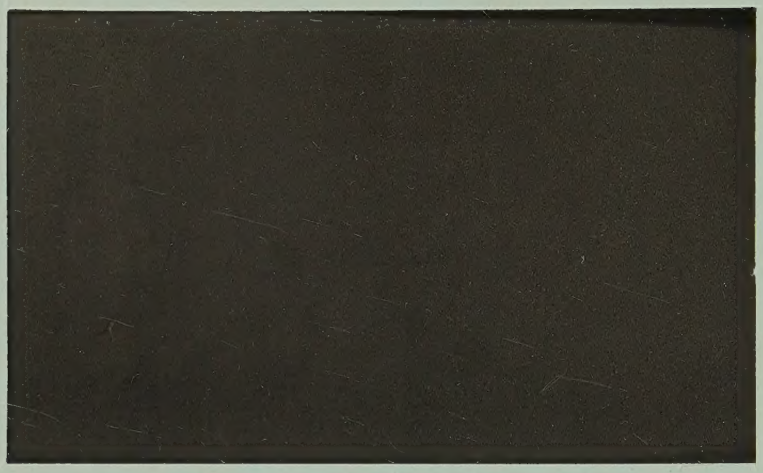

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SPECTRAL THEORY OF SYMMETRIC ORDINARY DIFFERENTIAL OPERATORS WITH AN INDEFINITE WEIGHT FUNCTION

By

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Abstract:

In this paper special properties of the singular differential equation

$$\sum_{k=0}^{n-1} (-1)^{n-k} (p_k y^{(n-k)})^{(n-k)} = \lambda r y \quad \text{on} \quad [0,b) , \qquad (*)$$

$0 < b \leq \infty$, with some boundary condition at 0 are studied in case that r is indefinite but the functions p_k are nonnegative. Extensive use is made of the theory of definitizable operators in the Krein space L^2_r, equipped with the indefinite scalar product $[f,g] = \int f g r \, dx$, integration over $[0,b)$, and the properties of functions of class $N_\kappa(\mathbb{C}^n)$ especially if $\kappa = 1$.

In case of a second order equation, that is $n = 1$ in $(*)$, a general expansion theorem is proved, extending classical results of H. Weyl and others to arbitrary selfadjoint boundary conditions at 0. The essential feature is the appearance of certain singularities of the spectral function or of nonreal eigenvalues. In the case of equation $(*)$ a generalization of the Sturm-Liouville boundary conditions is given and it is shown that some "coefficient of Weyl" can be introduced, belonging to a certain class $N_\kappa(\mathbb{C}^n)$. Using the method of directing functionals, it is shown, that there exists a spectral function, such that some kind of Parseval's relation holds true.

Key words and phrases: ordinary differential equation, spectral function, spaces with indefinite metric, indefinite weight function, definitizable operators.

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For technical reasons Chapters 1, 2 and 3 are contained in Part I and Chapter 4, 5 and References in Part II.

0. INTRODUCTION

In this paper we consider a differential equation of the form:

$$M(y) \equiv \sum_{k=0}^{n-1} (-1)^{n-k} (p_k (y^{(n-k)})^{(n-k)})(x) = \lambda r(x) y(x), \quad 0 \le x < b, \quad (0.1)$$

where $0 < b \le \infty$. If the so called "weight function" r is positive, under certain conditions on the coefficients p_k, $k = 0,1,\ldots,n$, and r, this equation can be studied in the context of the Hilbert space $L^2_r[0,b)$. In this case, generally, no restrictions about the signs of the p_k, $k = 0,1,\ldots,n$, are needed (see e.g. Jörgens [12] for the case $n = 1$, K. Kodaira [23] and M.A. Naimark [30] for the case of arbitrary n, $r(x) \equiv 1$; the study of the second-order case $n = 1$ goes back to the pioneering papers by H. Weyl [40], [42], see also E.C. Titchmarsh [39]).

Already in 1910, H. Weyl [41] treated for $n = 1$ also the case when the function r changes its sign. More explicitely, he studied the initial value problem

$$-(p_0 y')' + p_1 y = \lambda r y \quad \text{on} \quad [0,b) \quad (0.2)$$

$$y(0) \cos\alpha - p_{(0)} y'(0) \sin\alpha = 0, \quad \alpha \in [0,\pi) \quad (0.3)$$

assuming that $\alpha = 0$, $p_0 > 0$, $p_1 \ge 0$ a.e. on $[0,b)$, but without any restriction about the sign of r.

We mention, that (0.1) and (0.2) with appropriate boundary conditions are special cases of the more general problem $Su = \lambda Tu$ with two operators S, T. This equation has been studied in recent years by Å. Pleijel et al. in case that one of the operators S, T corresponds to a positive definite scalar product. This includes the original investigations by H. Weyl as well as other results of the above mentioned authors.

In case of the problem (0.2 - 3), e.g., as a scalar product connected with the right hand side,

$$[f,g] = \int_0^b f(x)\overline{g(x)}r(x)dx \quad (0.4)$$

can be chosen (for suitable functions f,g), which is not definite if

r changes its sign. As a scalar product connected with the left hand expression of (0.2 - 3) one chooses the Dirichlet form

$$\begin{cases} \int_0^b (p_0(x)f'(x)\overline{g'(x)} + p_1(x)f(x)\overline{g(x)})dx & \text{if } \alpha = 0, \\ \int_0^b (p_0(x)f'(x)\overline{g'(x)} + p_1(x)f(x)\overline{g(x)})dx + f(0)\overline{g(0)}\cot\alpha & \text{if } 0<\alpha<\pi, \end{cases} \tag{0.5}$$

which is evidently non-negative in Weyl's case, that is $\alpha = 0$, $p_0 > 0$, $p_1 \geq 0$. If, however, α is close to π, the form (0.5) becomes indefinite. Hence, if also r is indefinite, neither of the forms (0.4) and (0.5) is positive definite. This general case has been treated recently in a paper by F.V. Atkinson, W.N. Everitt and K.S. Ong [3]. These authors introduce Weyl's m-function and show that for sufficiently large α in (0.5) it can have a non-real singularity in each of the (upper and lower) half-planes $\mathbb{C}_+$ and $\mathbb{C}_-$. We mention, that some results in this "indefinite" case had also been obtained by H. Langer in [28].

In [6], H. Langer and the author studied the problem (0.2 - 3), with indefinite r and for general $\alpha \in [0,\pi)$. As an underlying space the Krein space L^2_r with positive definite scalar product:

$$(f,g) = \int_0^b f(x)\overline{g(x)}\,|r(x)|\,dx \tag{0.6}$$

and with indefinite scalar product (0.4) has been used. This study was mainly based on the observation, that although both forms (0.4) and (0.5) are indefinite if α is large, the form (0.5) has at most one negative square (that is, it can be negative definite only on an at most one-dimensional subspace). It was shown that some function, connected closely with the m-function of Weyl, belongs to the class $N_1(\mathbb{C})$, introduced by M.G. Krein and H. Langer in [21]. This function class is an analogue of the well-known class $N_0(\mathbb{C})$ of Nevanlinna functions (mapping $\mathbb{C}_+$ holomorphically into itself), which plays an important rôle in the study of the problem (0.2 - 3) for a positive function r, see e.g. E.C. Titchmarsh [39].

In this paper we present the results for the second-order case of [6] and generalize some of them to the 2n-th order case. More explicitly, we consider the equation (0.1), when r can change its sign, but $p_0 > 0$, $p_j \geq 0$, $j = 1,\dots,n$, a.e. on $[0,b)$ and with the initial condition:

$$(\cos\alpha)\,\hat{y}_1(0) - (\sin\alpha)\,\tilde{I}_n\hat{y}_2(0) = 0. \tag{0.7}$$

Here α is a real symmetric $(n\times n)$-matrix with all eigenvalues in $[0,\pi)$, $\hat{y}_1$ is the column of the derivatives of y of orders $0,1,\dots,n-1$, $\hat{y}_2$ is the column of the so called quasi-derivatives of y of orders from n to $2n-1$ and $\tilde{I}_n$ is the $(n\times n)$-matrix $\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$ (see Section 2.3).

The crucial fact encountered in the investigation is that the forms corresponding to (0.4) and (0.5) are generally indefinite. The form corresponding to (0.5), however, has at most n negative squares. Instead of the scalar functions of class $N_1(\mathbb{C})$, here the $(n\times n)$-matrix functions of class $N_\kappa(\mathbb{C}^n)$, introduced by M.G. Krein and H. Langer in [22], are involved.

A brief list of the contents of the thesis runs as follows.

Chapter 1 contains a short survey of the theory of Krein and Pontrjagin spaces as well as some facts about the functions of class $N_\kappa(\mathbb{C})$.

The second chapter is devoted to the study of the equation

$$M(y) = |r|f \tag{0.8}$$

in the Hilbert space $L^2_{|r|}(I)$ $(I = [0,b),\ 0 < b \leq \infty)$ with scalar product (0.6). In Section 2.1 a quasi-differential expression $M(y)$ of order $2n$ is introduced according to [30]. There M.A. Naimark treats the equation $M(y) = f$, which evidently defines an operator in $L^2(I)$. However, if $r(x) \neq 1$, this is not necessarily so in $L^2_{|r|}(I)$. Therefore, in Section 2.2 we formulate certain basic assumptions under which the equation (0.8) defines a (maximal) operator $\tilde{B}$: $\tilde{B}y = f$ in $L^2_{|r|}(I)$ (see Section 2.5). In Section 2.6 a generalization of the classical Sturm-Liouville boundary conditions is given for the equation (0.8) if $n > 1$. These boundary conditions determine the symmetric restrictions of $\tilde{B}$. An analogue of

Weyl's m-function for the case $n > 1$ is introduced in Section 2.7. In Section 2.8 the resolvent is expressed as an integral operator. Finally, more complete results in the second-order case are given in Section 2.9.

The chapters 3-5 contain the main results of the dissertation. In chapter 3, certain operators in the Krein space $L^2_r(I)$ are associated with the equation:

$$M(y) = rf. \tag{0.9}$$

In Section 3.1 we introduce the important notion of r-limit (n,n) case for the singular boundary point b, which plays the same role here as the notion of limit point case in H. Weyl's theory. After the more technical results of Section 3.2, in Section 3.3 the number of negative squares of the form corresponding to (0.5) is estimated (Theorem 3.3.4). In Section 3.4 the $(n \times n)$-matrix function $W(\lambda)$ which generalizes Weyl's m-function to the case considered here, is introduced. Theorem 3.4.2 gives its existence and uniqueness as well as the representation of the associated resolvent. In Theorem 3.4.4 it is shown that the function

$$\hat{W}(\lambda) = \lambda(W(\lambda) + (\cot \alpha)) \qquad (0 \leq \alpha < \pi I_n)$$

belongs to some class $N_\kappa(\mathbb{C}^n)$.

In chapter 4, Theorem 4.3.2, a Parseval's relation is given for the equation (0.9), based on a generalization of M.G. Krein's method of directing functionals (we call these Krein transformation). This is an adaption of certain unpublished results of M.G. Krein and H. Langer [17].

Chapter 5 concerns the second-order case. The main result is a general expansion theorem (Theorem 5.5.1) for the equation (0.9) in the r-limit (1,1) case at b. The new feature here (compared with the case of a positive function r) is the appearance of certain singularities of the spectral function as well as the possible existence of a non-real pair of eigenvalues.

In connection with the results of this paper some problems are still

open. E.g., it would be of interest to generalize the expansion theorems of Chapter 5 to the case of an equation of arbitrary order. This depends on a suitable representation theorem for functions of class $N_K(\mathfrak{T}^n)$, which is not yet available. Also in the special case, when the spectrum of the differential operator is discrete, the expansion theorem should be completed by giving conditions for uniform convergence etc.

It can be expected, that similar results hold in case that the defect index of the considered minimal operator is greater than n (with suitable conditions at the right endpoint imposed) or both endpoints are singular.

Finally, the basic assumptions I - IV can perhaps be relaxed by using the theory of linear relations instead of operators. This should make it possible to study the more general problem $Su = \lambda Tu$ where the forms associated with S and T are both indefinite but one of these forms has only a finite number of negative squares, in the spirit of the papers of Åke Pleijel.

1. GEOMETRICAL PRELIMINARIES

1.1. Krein spaces, Pontrjagin spaces

In this paper we consider a Hilbert space H, where in addition to the usual scalar product $(\cdot,\cdot)$, an indefinite scalar product $[\cdot,\cdot]$ is defined with the properties (i) - (iv):

(i) $[f,g] = \overline{[g,f]}$, $f,g \in H$.

(ii) $[\cdot,\cdot]$ is linear in its first argument.

It follows that $[\cdot,\cdot]$ is conjugate linear in its second argument.

(iii) The clearly real value $[f,f]$ takes both positive and negative signs.

(iv) $[f,g]$ is a continuous function of f and g in the metric of the Hilbert space.

These properties imply, by the wellknown representation theorem of M. Fréchet and F. Riesz, the existence of a bounded selfadjoint operator G such that

$$[f,g] = (Gf, g) \quad (f,g \in H). \tag{1.1.1}$$

If $G = J$, where J has the special properties $J = J^*$, $J^2 = I$, then H is called a Krein space (see Bognár [4]). We note that the scalar product $(\cdot,\cdot)$ determines the topology in H, whereas in the following the direction of our investigations is connected with the indefinite scalar product.

Let H be a Krein space with the indefinite scalar product $[f,g]$. It has (infinitely many) decompositions

$$H = H_+ \,[\dot{+}]\, H_- , \tag{1.1.2}$$

such that H_+ (H_-) is a Hilbert space with respect to the scalar product $[\cdot,\cdot]$ ($-[\cdot,\cdot]$ resp.). Here $[\dot{+}]$ denotes a direct sum, orthogonal with respect to $[\cdot,\cdot]$, so that

$$H_+ \cap H_- = \{0\}\,,\quad [f_+,f_-] = 0 \quad \text{when} \quad f_+ \in H_+,\ f_- \in H_-\,.$$

A decomposition (1.1.2) gives rise to a positive definite scalar product

$$(f,g)' = [f_+,g_+] - [f_-,g_-] \quad \text{if} \quad f = f_+ + f_-,\ g = g_+ + g_- \quad (f_\pm \in H_\pm). \tag{1.1.3}$$

If $P_\pm$ denote the orthogonal projectors of H on $H_\pm$,

$$[f,g] = (Jf,g) \quad \text{where} \quad J = P_+ - P_-\,,\ P_+ + P_- = I\,.$$

Different decompositions (1.1.2) generate different scalar products (1.1.3) but the corresponding norms are all equivalent (see Bognár [4]).

Put

$$\mathcal{P}_+ = \{f:\ [f,f] \geq 0\}\,,\ \mathcal{P}_- = \{f:\ [f,f] \leq 0\}\,,\ \mathcal{P}_0 = \{f:\ [f,f] = 0\}\,.$$

A subspace L of H, belonging to $\mathcal{P}_+$, is called nonnegative, it is called positive if $f \in L$, $f \neq 0$ implies $[f,f] > 0$, and uniformly positive, if there exists a constant $\gamma > 0$, such that $[f,f] \geq \gamma(f,f)$ for all $f \in L$. Nonpositive, negative, uniformly negative and neutral subspaces are similarly defined.

A subspace $L \subset \mathcal{P}_+$ is said to be maximal if it is not a proper part of any other nonnegative subspace of H, i.e. if $L \subset L_1 \subset \mathcal{P}_+$ implies $L_1 = L$.

If in the representation (1.1.2) $\kappa = \min(\dim H_+, \dim H_-) < \infty$, then H is called a Pontrjagin space of index κ and is often denoted by Π_κ. In case $\dim H_+ = \kappa_+$ (resp. $\dim H_- = \kappa_-$) is finite, we shall say that H is a Pontrjagin space with κ_+ positive (resp. κ_- negative) squares.

A set L is called a pre-Pontrjagin space with κ negative squares, when the following conditions (i) - (iv) are satisfied:

(i) L is a complex linear space.

(ii) L is provided with a hermitian form $[\cdot,\cdot]$ [1].

(iii) On a κ-dimensional subspace N, the restriction of $[\cdot,\cdot]$ is negative.

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[1] that is, $[\cdot,\cdot]$ is a mapping from $L \times L$ in $\mathbb{C}$ satisfying (i) and (ii) on p. 1.1.1.

(iv) There is no $(\kappa+1)$-dimensional subspace $\tilde{N}$ of L, such that the restriction of $[\cdot,\cdot]$ to $\tilde{N}$ is negative.

The conditions (iii) and (iv) mean that for arbitrary elements $f_1, f_2, \ldots, f_{\kappa+1}$ of L the hermitian matrix $[f_j,f_k]_{j,k=1}^{\kappa+1}$ has no more than κ negative eigenvalues, and has exactly κ for at least one choice of $f_1,\ldots,f_{\kappa+1} \in L$. This is expressed by saying that the <u>indefinite scalar product</u> $[\cdot,\cdot]$ <u>has</u> κ <u>negative squares</u>.

Let r be a measurable function on $[0,b)$, $0 < b \leq \infty$, summable on each closed subinterval of $[0,b)$. The Krein space we deal with in the following is the space $H = L_r^2$ consisting of all complex "functions" f on $[0,b)$ with the property $\int_0^b |r(x)||f(x)|^2 dx < \infty$. In $H = L_r^2$

$$(f,g) = \int_0^b |r(x)|\, f(x)\overline{g(x)}dx\,, \tag{1.1.4}$$

$$[f,g] = \int_0^b r(x)f(x)\overline{g(x)}dx\,. \tag{1.1.5}$$

In this case

$$(Ju)(x) = (\operatorname{sign} r(x))u(x)\,,\quad (P_\pm u)(x) = \tfrac{1}{2}(1 \pm \operatorname{sign} r(x))u(x) \quad \text{a.e. on } x \in [0,b)\,,\quad u \in L_r^2\,. \tag{1.1.6}$$

Also $H_\pm = P_\pm L_r^2$.

1.2. <u>J-selfadjoint operators</u>

Let A be a densely defined linear operator in the Krein space H. Then the J-adjoint A^+ of A is determined by the relation

$$[Af,g] = [f,A^+g]\,,\quad f \in D(A)\,,\quad g \in D(A^+)\,,$$

where $D(A)$ and $D(A^+)$ denote the domains of A and A^+. More exactly, $D(A^+)$ is the set of every g such that the scalar product $[Af,g]$ has a representation $[f,h]$ with some element $h = A^+g \in H$. If $B = JA$, then

$$A^+ = JB^*J\,,$$

where B^* is the Hilbert space adjoint of B.

The operator A is called J-symmetric if $A \subset A^+$, and J-selfadjoint if $A = A^+$.

Spectrum $\sigma(A)$, point spectrum $\sigma_p(A)$, continuous spectrum $\sigma_c(A)$, residual spectrum $\sigma_r(A)$ and resolvent set $\rho(A)$ are defined as in Bognár [4].

THEOREM 1.2.1. (H. Langer [24] or Bognár [4]). Let A be a J-selfadjoint operator in a Krein space H. Then

(i) $\sigma(A)$ is symmetric with respect to the real axis.

(ii) If $\lambda \in \sigma_p(A)$ then $\bar{\lambda} \in \sigma_p(A) \cup \sigma_r(A)$.

(iii) If $\lambda \in \sigma_r(A)$ then $\bar{\lambda} \in \sigma_p(A)$.

COROLLARY 1.2.2.

(i) The set $\sigma_p(A) \cup \sigma_r(A)$ lies symmetric with respect to the real axis.

(ii) No real point belongs to $\sigma_r(A)$.

(iii) $\sigma_c(A)$ lies symmetric with respect to the real axis.

Two subspaces H_1 and H_2 of the Krein space H are called skewly linked, if for every $f \in H_1$ there exists a $g \in H_2$ and for every $g \in H_2$ an $f \in H_1$ such that in both cases $[f,g] = 1$.

Let G_λ be the algebraic eigenspace belonging to the possible eigenvalue λ of A, i.e. $G_\lambda = \{f: (A - \lambda I)^k f = 0$ for some natural number $k\}$.

PROPOSITION 1.2.3. (Compare Bognár [4]). If A is J-symmetric in the Krein space H, and if $\lambda, \mu \in \sigma(A)$, $\lambda \neq \bar{\mu}$ then $G_\lambda \,[\perp]\, G_\mu$ [1]. In particular $G_\lambda \subset \mathcal{P}_0$ if $\lambda \neq \bar{\lambda}$.

Proof. We have to show that the relations $(A - \lambda I)^n f = 0$, $(A - \mu I)^m g = 0$ imply $[f,g] = 0$. In case $n + m = 0$ (or more general, in case $n = 0$ or $m = 0$) this is obvious. Assume that it is proved for arbitrary exponents

---

[1] $[\perp]$ denotes orthogonality with respect to the indefinite scalar product $[\cdot,\cdot]$.

with sum less than $n+m$, and put $(A-\lambda I)f = f_1$, $(A-\mu I)g = g_1$. Then

$$(A-\lambda I)^{n-1} f_1 = 0, \quad (A-\mu I)^m g = 0,$$
$$(A-\lambda I)^n f = 0, \quad (A-\mu I)^{m-1} g_1 = 0,$$

which by assumption implies that $[f_1,g] = [f,g_1] = 0$. Therefore,

$$\lambda[f,g] = [Af - f_1,g] = [f,Ag] = \bar{\mu}[f,g],$$

and the proposition follows.

PROPOSITION 1.2.4. (Bognár [4]). Let $A = A^+$, and let $\lambda \in \sigma_p(A)$ be a non-real isolated point of $\sigma(A)$ with $\dim G_\lambda = n < \infty$. Then $\bar{\lambda} \in \sigma_p(A)$, $\dim G_{\bar{\lambda}} = n$. The spaces G_λ and $G_{\bar{\lambda}}$ are skewly linked: for each basis $f_1,\dots,f_n$ in G_λ there is a basis $g_1,\dots,g_n$ in $G_{\bar{\lambda}}$ such that $[f_j,g_k] = \delta_{jk}$ $(1 \le j, k \le n)$. For the basis $f_1,\dots,f_n$, $g_1,\dots,g_n$ of $G_\lambda [\dot{+}] G_{\bar{\lambda}}$ the restriction of A to this subspace has the matrix representation

$$\begin{pmatrix} S & 0 \\ 0 & S^* \end{pmatrix}$$

where S^* denotes the adjoint matrix of S. In particular, elementary divisors of $A|_{G_\lambda}$ and $A|_{G_{\bar{\lambda}}}$ are in one-to-one correspondence (orders inclusive).

The J-selfadjoint operator A is called J-nonnegative if $[Af,f] \ge 0$ $(f \in D(A))$. Moreover, if we suppose $\rho(A) \neq \emptyset$ then (as a consequence of Theorem 1.3, VII of Bognár [4]), $\sigma(A)$ is real and by our Corollary 1.2, $\sigma_r(A) = \emptyset$.

1.3. Definitizable operators

A J-selfadjoint operator A in the Krein space H is called definitizable, if there exists a polynomial P such that

$$[P(A)f,f] \ge 0 \quad (f \in D(P(A))). \tag{1.3.1}$$

(In Bognár [4] bounded operators with this property are called positizable.)

If in addition $\rho(A) \neq \emptyset$, then the spectral mapping theorem and the fact, that the spectrum of a J-nonnegative operator is real, imply that the values of the polynomial P on $\sigma(A)$ are real. Moreover, the non-real spectrum of A consists of a finite number of normal (Bognár [4]) eigenvalues which are zeros of P. The span L of the corresponding algebraic eigenspaces is orthocomplemented, i.e., if

$$L^{[\perp]} = \{f: [f,L] = \{0\}\}, \tag{1.3.2}$$

then $H = L \,[\dot{+}]\, L^{[\perp]}$. It follows that the restriction of A to $L^{[\perp]}$ is again definitizable and its spectrum is real.

PROPOSITION 1.3.1. (Compare with Bognár [4], § IX, Theorem 7.3). *In a Pontrjagin space Π_κ, every J-selfadjoint operator A is definitizable. The definitizing polynomial P can be chosen to have a degree not greater than 2κ.*

Proof. By a theorem of L.S. Pontrjagin (Theorem 15 in [12]), there exists a κ-dimensional J-nonpositive subspace L, which is invariant under A. If Q denotes the minimal polynomial of the restriction of A to L, then $Q(A)L = \{0\}$. This yields $\overline{Q}(A)\Pi_\kappa \,[\perp]\, L$. Therefore, $\overline{Q}(A)\Pi_\kappa \subset \mathcal{P}_+$ and (1.3.1) is obtained with $P = Q\overline{Q}$.

A definitizable operator A with $\rho(A) \neq \emptyset$ admits an *eigenspectral function* with critical points. In order to explain this, consider a finite set

$$S = \{s_1, s_2, \ldots, s_n\} \subset \overline{\mathbb{R}}$$

($\overline{\mathbb{R}}$ is the one point compactification of the real line $\mathbb{R}$), and denote by $\mathbb{R}_S$ the algebra of Borel subsets Δ of $\overline{\mathbb{R}}$ with the property $(\overline{\Delta} \setminus \Delta) \cap S = \emptyset$. (The closure of a set M in $\overline{\mathbb{R}}$ is $\overline{M}$.)

A mapping E from $\mathbb{R}_S$ into the set of J-selfadjoint projectors in H is called an *eigenspectral function of the definitizable operator A with critical points* $s_1, s_2, \ldots, s_n$, if the following conditions (1) - (6) are satisfied ($\Delta, \Delta_1, \Delta_2 \in \mathbb{R}_S$):

(1) $E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2)$ if $\Delta_1 \cap \Delta_2 = \emptyset$.

(2) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)\, E(\Delta_2)$.

(3) $E(\emptyset) = 0$, $E(\overline{\mathbb{R}})\, H = L^{[1]}$ (see 1.3.2).

(4) $\Delta \cap S \neq \emptyset$ implies that the scalar product $[\cdot,\cdot]$ is indefinite on $E(\Delta)\, H$.

(5) $E(\Delta)\, A = A\, E(\Delta)$.

(6) $\sigma(A \mid E(\Delta)\, H \subset \overline{\Delta}$, where $\sigma(A \mid E(\Delta)\, H$ denotes the spectrum of the restriction of A to $A(\Delta)\, H$.

Let A be a definitizable J-selfadjoint operator with $\rho(A) \neq \emptyset$ and P a definitizing polynomial. Then A admits an eigenspectral function E with a set S of critical points:

$$S \subset \{x \in \mathbb{R}: P(x) = 0\} \cup \{\infty\}.$$

Here ∞ can be a critical point only if P is of odd degree. Moreover,

(7) $[E(\Delta)f,f] \geq 0$ (≤ 0) if $P(t) > 0$ $(< 0$ resp.)
for $t \in \Delta$, $\Delta \in \mathbb{R}_S$.

The critical points of the eigenspectral function of A are also called critical points of A. It can be shown that a J-selfadjoint operator A has at most one eigenspectral function (H. Langer [29]).

Let $\Delta = (a,b) \in \mathbb{R}_S$ be an open bounded interval. Then the following formula holds true:

$$E(\Delta) + \frac{1}{2} E(\overline{\Delta} \setminus \Delta) = -\frac{1}{2\pi i} \int'_{\Gamma(\Delta;\varepsilon)} (A - zI)^{-1} dz = F(\Delta).$$

Here $\Gamma(\Delta;\varepsilon)$ is the contour $a + i\varepsilon$, $b + i\varepsilon$, $b - i\varepsilon$, $a - i\varepsilon$, $a + i\varepsilon$, $\varepsilon > 0$ sufficiently small, the prime indicates that Cauchy principal values are taken in a and b. Moreover, if $(\Delta_n) \subset \mathbb{R}_S$ is an increasing sequence of open intervals $(\Delta_n \subset \Delta_{n+1})$ and $\Delta_n \uparrow \Delta \in \mathbb{R}_S$, we have

$$E(\Delta) = \lim_{n \to \infty} F(\Delta_n)$$

in the strong operator topology.

A critical point s of A is said to be regular, if for an open interval Δ_s, $s \in \Delta_s$, with positive distance from all the other critical points of A, the set

$$\{\|E(\Delta)\| : \Delta \in R_S,\ \Delta \subset \Delta_s\}$$

is bounded. Otherwise s is called singular.

If s is an eigenvalue and a regular critical point of A, $E(\Delta_s)H$ is decomposed according to

$$E(\Delta_s)H = G_s \,[\dot{+}]\, G_s^{[\perp]} \tag{1.3.3}$$

where G_s is the algebraic eigenspace belonging to s, and the restriction $A|_{G_s^{[\perp]}}$ is selfadjoint in $G_s^{[\perp]}$ with respect to $|\cdot,\cdot|$.

For a J-selfadjoint and J-nonnegative operator A, the definitizing polynomial P is $P(t) = t$. If in addition $\rho(A) \neq \emptyset$, the spectrum $\sigma(A)$ is real, and only 0 and ∞ can be critical points of A: $S \subset \{0,\infty\}$.

Consider a neighbourhood Δ_∞ of ∞ (that is Δ_∞ contains the complement of a compact set of $\mathbb{R}$). Then

$$Au = \int_{\mathbb{R}\setminus\Delta_\infty} \lambda E(d\lambda)u + Nu \quad \text{for} \quad u \in (I - E(\Delta_\infty))H,$$ [1)]

where N is a bounded J-nonnegative operator with the properties $N^2 = 0$, $NE(\Delta) = E(\Delta)N = 0$ $(\Delta \in R_S,\ 0 \notin \Delta)$. Moreover, if 0 is not an eigenvalue of A, then $N = 0$.

LEMMA 1.3.2. If A is a J-nonnegative J-selfadjoint operator with $\rho(A) \neq 0$, and the neighbourhood Δ_∞ of ∞ has a positive distance from zero, then $u \in D(A)$ implies $E(\Delta_\infty)u = \int_{\Delta_\infty} E(d\lambda)u$, where E denotes the eigenspectral function of A.

The integral on the right hand side is to be understood as an improper integral with respect to the singularity of E at ∞. It exists in the strong operator topology.

---

1) The integral exists as an improper integral at 0 in the norm topology of H.

Proof. It is sufficient to consider the restriction of A to $E(\Delta_\infty)H$, that is, we may suppose $\sigma(A) \subset \Delta_\infty$. The bounded inverse $A^{-1} = T$ is J-nonnegative, and $u \in D(A)$ implies $u = Tv$ with a certain $v \in H$. Since zero is not an eigenvalue of T,

$$u = Tv = \int_{\sigma(T)} \mu F(d\mu) v = \int_{\sigma(A)} \lambda^{-1} E(d\lambda) v = \int_{\sigma(A)} \lambda^{-1} E(d\lambda) Au = \int_{\sigma(A)} E(d\lambda) u. \quad (1.3.4)$$

Here F denotes the eigenspectral function of T and use has been made of the relation

$$F(\Delta') = E(\Delta) \quad \text{if} \quad \Delta' = \{t^{-1} : t \in A\}, \ 0 \notin \Delta' \cup \Delta.$$

The integrals in (1.3.4) over $\sigma(T)$ and $\sigma(A)$ exist as improper integrals in the norm of H with respect to the singularity at 0 and ∞ resp.

1.4. Index and order of a critical point

DEFINITION 1.4.1. (See M.G. Krein, H. Langer [19], H. Langer [20].) *A critical point s is said to have a finite index $\kappa(s)$, if there exists an open interval $\Delta \in R_s'$, $\Delta \cap S = \{s\}$ such that $E(\Delta)H$ is a space $\Pi_{\kappa(s)}$.*

The order $q(s)$ of an isolated critical point s of the J-eigenspectral function E is the smallest nonnegative integer $q(s)$ for which the integral

$$\int_{\Delta_s} |t-s|^{2q(s)} E(dt) =$$

$$= \lim_{\varepsilon_1, \varepsilon_2 \to \infty} \left\{ \int_{[-\infty, s-\varepsilon_1) \cap \Delta_s} + \int_{(s+\varepsilon_2, \infty) \cap \Delta_s} \right\} (t-s)^{2q(s)} E(dt)$$

exists. Here Δ_s is an open neighbourhood of s.

The order $q(s)$ of the critical point s is zero if and only if s is regular.

Let G_s be the algebraic eigenspace belonging to the eigenvalue s and put $F_s = G_s \cap G_s^{[\perp]}$.

THEOREM 1.4.2. (M.G. Krein, H. Langer [19], H. Langer [29].) Assume that the J-selfadjoint operator A is definitizable, $\rho(A) \neq \emptyset$ and the critical point s is of finite index $\kappa(s)$ and order $\eta(s)$. Then $(A - sI)^{\nu(s)} G_s = \{0\}$, where $\nu(s) = 2\kappa(s) + 1 - (2p(s) - \eta(s))$ and $p(s) = \dim F_s$.

1.5. Functions of class N_κ.

Let κ be an integer ≥ 0. Denote by N_κ the set of meromorphic functions Q on $\mathbb{C}_+ = \{z = x + iy: -\infty < x < \infty, y > 0\}$ with the property that the kernel

$$N_Q(z,\zeta) = (Q(z) - \overline{Q(\zeta)})\,(z - \overline{\zeta})^{-1}, \quad z,\zeta \in Z_Q,$$

has exactly κ negative squares, by which is meant that every quadratic form

$$\sum_{i,j=1}^{n} N_Q(z_i,z_j)\,\xi_i\overline{\xi_j}$$

for arbitrary $z_1, z_2, \ldots, z_n$ in the holomorphy domain Z_Q of Q has at most κ negative squares, and at least one such form has exactly κ negative squares. Every function $Q \in N_\kappa$ is extended to $\mathbb{C}_+ \cup \mathbb{C}_-$ by the relation $Q(\overline{z}) = \overline{Q(z)}$. Then the extended Z_Q contains the halfplanes $\mathbb{C}_+$ and $\mathbb{C}_-$ with possible exception of at most κ pairs of points symmetric to the real axis. If we define $N_Q(z,\overline{z}) = \frac{dQ(z)}{dz}$, the kernel N_Q has still κ negative squares on the holomorphy domain of the extended function Q.

In M.G. Krein, H. Langer [21], §3, it was shown that a function Q belongs to N_κ, $\kappa \geq 0$, if and only if it has a representation

$$Q(z) = \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \sum_{j=0}^{r} S_j(t,z)\right)(1+t^2)^{1+\rho_0} \prod_{j=1}^{r} \frac{(1+t^2)^{\rho_j}}{(t-t_j)^{2\rho_j}}\, d\sigma(t) + \tag{1.5.1}$$

$$+ \hat{R}_0(z) + \sum_{j=1}^{r} R_j\left(\frac{1}{t_j - z}\right) + \sum_{k=1}^{s} \left(T_k\left(\frac{1}{z-\beta_k}\right) + \overline{T}_k\left(\frac{1}{z-\beta_k}\right)\right),$$

where

a) $r \geq 0$, $s \geq 0$, $t_1,\ldots,t_r$ are different real, $\beta_1,\ldots,\beta_s$ different non-real numbers with $\operatorname{Im} \beta_k > 0$. If $r=0$ $(s=0)$, no points $t_j(\beta_k)$ appear in (1.5.1) and the corresponding sums are replaced by 0 (the corresponding product by 1).

b) σ is a left continuous, nondecreasing, bounded function on the real axis, continuous at $t_1,\ldots,t_r$.

c) Let $\Delta_1,\ldots,\Delta_r,\Delta_\infty$ be mutually disjoint neighbourhoods of $t_1,\ldots,t_r,\infty$ resp., then

$$S_0(t,z) = \begin{cases} (t+z)\sum_{\nu=1}^{\rho_0+1} (1+z^2)^{\nu-1}(1+t^2)^{-\nu} & \text{if } t \in \Delta_\infty, \\ 0 & \text{if } t \notin \Delta_\infty, \end{cases}$$

$$S_j(t,z) = \begin{cases} -\sum_{\nu=1}^{2\rho_j} (t-t_j)^{\nu-1}(z-t_j)^{-\nu} & \text{if } t \in \Delta_j,\ j=1,\ldots,r \\ 0 & \text{if } t \notin \Delta_j. \end{cases}$$

d) $\hat{R}_0$, R_j are polynomials with real coefficients such that $R_j(0) = 0$, $j = 1,2,\ldots,r$.

e) T_k are polynomials of degree τ_k (>0) such that $T_k(0) = 0$, $k = 1,2,\ldots,s$.

f) $\rho_j > 0$ implies $\int_{-\infty}^{\infty} (t-t_j)^{-2}\, d\sigma(t) = \infty$, $j = 1,\ldots,r$, and

$\rho_0 > 0$ implies $\int_{-\infty}^{\infty} t^2\, d\sigma(t) = \infty$.

g) $\rho_j = 0$ implies that the corresponding polynomial $R_j(z)$ does not have the form $c_j z$ with a nonnegative constant c_j, $j = 1,\ldots,r$.

The representation (1.5.1) of $Q \in N_\kappa$ with the properties a)-g) is uniquely determined when $\Delta_1, \Delta_2, \ldots, \Delta_r, \Delta_\infty$ are fixed, and

$$\kappa = \sum_{k=1}^{s} \tau_k + \sum_{j=0}^{r} \kappa_j, \quad \kappa_j = \max(\rho_j, [(r_j+1)/2]) - \delta_j \qquad (1.5.2)$$

In (1.5.2) r_j is the degree of R_j, $j=1,2,\ldots,r$, and r_0 is the degree of

$$R_0(z) = \hat{R}_0(z) - z(1+z^2)^{\rho_0} \int_{\Delta_\infty} \prod_{j=1}^{r} \frac{(1+t^2)^{\rho_j}}{(t-t)^{2\rho_j}}\, d\sigma(t),$$

and $\delta_j = 1$ if $r_j > 2\rho_j$, r_j odd and the highest coefficient of R_j is positive, otherwise $\delta_j = 0$, $j = 0,1,2,\ldots,r$. If $r > 0$, the points t_j, $j = 1,2,\ldots,r$, in the representation (1.5.1) will be called <u>critical points</u> of the function $Q \in N_\kappa$; ∞ is called a <u>critical point</u> of Q if $\kappa_0 > 0$.

The following lemma deals with two functions $Q \in N_\kappa$, $\tilde{Q} \in N_{\tilde{\kappa}}$. $\tilde{r}$, $\tilde{\rho}_j$ etc. are the quantities in the representation (1.5.1) of $\tilde{Q}$.

LEMMA 1.5.1. <u>Suppose</u> $Q \in N_\kappa$, $\tilde{Q} \in N_{\tilde{\kappa}}$ <u>and</u> $\tilde{\kappa}_0 = 0$ (which means that $\tilde{\rho}_0 = 0$, $\tilde{r}_0 \le 1$ and $\tilde{R}_0'(0) \ge 0$ in the representation (1.5.1) of $\tilde{Q}$). <u>Further, suppose that there exist two entire functions</u> p, q, $p(x) \ge 0$ <u>if</u> $x \in \mathbb{R}$, <u>such that</u>

$$\tilde{Q}(z) = Q(z)p(z) + q(z) \qquad (z \in Z_Q \cap Z_{\tilde{Q}}) \qquad (1.5.3)$$

<u>Then</u> $\tilde{\kappa} \le \kappa$.

<u>Proof</u>. Consider the representation (1.5.1) of Q and $\tilde{Q}$. (1.5.3) implies $\tilde{s} \le s$, and with a suitable enumeration $\beta_k = \tilde{\beta}_k$, $k = 1,\ldots,\tilde{s}$. Then $\tilde{\tau}_k \le \tau_k$.

Furthermore, from (1.5.3) and the Stieltjes-Livšic inversion formula (see Appendix) it follows that

$$(1+t^2)^{\tilde{\rho}_0} \prod_{j=1}^{\tilde{r}} (1+t^2)^{\tilde{\rho}_j} (t-\tilde{t}_j)^{-2\tilde{\rho}_j}\, d\tilde{\sigma}(t) =$$

$$= (1+t^2)^{\rho_0} \prod_{j=1}^{r} (1+t^2)^{\rho_j} \cdot (t-t_j)^{-2\rho_j} \, d\sigma(t)\, p(t), \quad t \neq t_j, \tilde{t}_j .$$

Therefore $\tilde{r} \leq r$, and if we assume $t_j = \tilde{t}_j$, $j = 1,\dots,\tilde{r}$,

$$\tilde{\rho}_j \leq \rho_j \quad \text{and} \quad \tilde{r}_j \leq r_j . \tag{1.5.4}$$

As p is nonnegative on $\mathbb{R}$, its zeros on $\mathbb{R}$ are of even order. Hence if for some $j = 1,2,\dots \tilde{r}$ we have $\tilde{r}_j < r_j$ and $\tilde{\rho}_j < \rho_j$, then $\tilde{r}_j \leq r_j - 2$, $\tilde{\rho}_j \leq \rho_j - 1$, and for the corresponding term in (1.5.2) we get $\tilde{\kappa}_j \leq \kappa_j$.

If the equality sign holds in one of the relations (1.5.4) we have $p(t_j) > 0$, therefore equality holds in both relations of (1.5.4), and $\delta_j = \tilde{\delta}_j$, that is $\tilde{\kappa}_j = \kappa_j$. Now, the conclusion follows from (1.5.2), as the inequality $\tilde{\kappa}_0 \leq \kappa_0$ is obvious.

REMARK 1.5.2. The preceeding results only dealt with the classes N_κ, $\kappa \geq 0$, of complex value functions. This definition can, however, be generalized to classes $N_\kappa(\mathbb{C}^n)$ of $(n \times n)$-matrix valued functions as-follows. If κ is a non-negative integer, denote by $N_\kappa(\mathbb{C}^n)$ the set of meromorphic $(n \times n)$-matrix functions Q on $\mathbb{C}_+$ with the property, that every quadratic form

$$\sum_{i,j=1}^{n} \left\langle \frac{Q(z_i) - Q(z_j)^*}{z_i - \overline{z}_j} x_i, x_j \right\rangle \xi_i \xi_j$$

for $z_1, z_2, \dots, z_n$ in the holomorphy domain of Q and $x_1, \dots, x_n \in \mathbb{C}^n$ has at most κ negative squares and at least one such form has exactly κ negative squares. Here $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{C}^n$.

2. QUASI-DIFFERENTIAL OPERATORS

2.1. Quasi-differential expression on $[0,b)$.

We consider formal differential expressions $M(y)$ on $[0,b)$ of order $2n$ given by

$$M(y) = \sum_{k=0}^{n} (-1)^{(n-k)} (p_k y^{(n-k)})^{(n-k)}, \quad y^{(j)} = d^j y/dx^j .$$

The coefficients p_k are always supposed to be real and $\frac{1}{p_0}, p_1, \ldots, p_n$ are assumed to be locally integrable on $[0,b)$. This assumption does not allow a direct definition of $M(y)$ even if all derivatives of y up to the order $2n$ exist. To give a meaning to $M(y)$, Naimark (see [20], p.48) defined quasi-derivatives $y^{[k]}$ of orders $k = 0,1,\ldots,2n$ by the formulae

$$\begin{aligned} y^{[0]} &= y , \\ y^{[k]} &= \frac{d}{dx} y^{[k-1]}, \quad k = 1,2,\ldots,n-1 , \\ y^{[n]} &= p_0 \frac{d}{dx} y^{[n-1]} , \\ y^{[n+k]} &= p_k y^{[n-k]} - \frac{d}{dx} y^{[n+k-1]}, \quad k = 1,2,\ldots,n . \end{aligned} \tag{2.1.1}$$

Then the appropriate definition is

$$M(y) = y^{[2n]} .$$

Evidently, the recursion formulae (2.1.1) define $y^{[2n]}$ a.e. as an integrable function provided the lower order quasi-derivatives up to (2n-1)-th order inclusive exist and are absolutely continuous over finite subintervals of $[0,b)$ (for short AC). In this case $y^{[0]}, y^{[1]}, \ldots, y^{[n-1]}$ coincide with the derivatives of the same orders (while $y^{[n]} = p_0 y^{(n)}$).

In case $y^{[2n]}$ exists as indicated we shall say that the expression $M(y)$ makes sense for the given function y.

It is important that partial integration formulae which can be expected due to the formal expression of $M(y)$ hold true with the expression $M(y) = y^{[2n]}$. This can be verified from the relation

$$\int_0^{b'} y^{[n+k]} \bar{v}^{(n-k)} dx = \int_0^{b'} (p_k y^{(n-k)} \bar{v}^{(n-k)} + y^{[n+k-1]} \bar{v}^{(n-(k-1))}) dx -$$

$$- \left[y^{[n+k-1]} \bar{v}^{(n-k)} \right]_0^{b'}$$

for $k = n, n-1, \ldots, 1$, where v is a given function for which the expression $M(v)$ makes sense and $[0,b']$ is a finite subinterval of $[0,b)$. Repeated application of the above partial integration formulae gives with $M(y) = y^{[2n]}$

$$\int_0^{b'} M(y) \bar{v} \, dx = \int_0^{b'} \sum_{k=0}^{n} p_k y^{(n-k)} \bar{v}^{(n-k)} dx - \sum_{k=1}^{n} \left[y^{[2n-k]} \bar{v}^{(k-1)} \right]_0^{b'} ,$$

and with $M(v) = v^{[2n]}$

$$\int_0^{b'} y \, \overline{M(v)} dx = \int_0^{b'} \sum_{k=0}^{n} p_k y^{(n-k)} \bar{v}^{(n-k)} dx - \sum_{k=1}^{n} \left[y^{(k-1)} \bar{v}^{[2n-k]} \right]_0^{b'} .$$

It follows that Lagrange's identity

$$M(y) \bar{v} - y \overline{M(v)} = \frac{d}{dx} h(y,v) ,$$
$$h(y,v) = \sum_{k=1}^{n} (y^{(k-1)} \bar{v}^{[2n-k]} - y^{[2n-k]} \bar{v}^{(k-1)}) , \qquad (2.1.2)$$

holds true if M makes sense for the functions y and v (Naimark [20], p.50).

2.2. Basic assumptions. Let $p_0, p_1, \ldots, p_n$ and r be real measurable functions defined a.e. on $[0,b)$, $0 \leq b < \infty$, satisfying the following assumptions:

I. $\frac{1}{p_0}, p_1, p_2, \ldots, p_n \in L_{loc}[0,b)$, and $r \in L_{loc}[0,b)$.

II. $p_0(x) > 0$ a.e. in $[0,b)$, $p_k(x) \geq 0$ a.e. in $[0,b)$, $k=1,2,\ldots,n$.

III. For arbitrary $x \in (0,b)$ we have

$$\int_0^x |r(t)|dt > 0\,, \quad 0 < \int_x^b |r(t)|dt \quad (\leq 0)\,,$$

that is $r(x)$ shall not vanish a.e. near 0 nor near b.

IV. There exists a sequence (b_m), $b_m \uparrow b$ $(m \to \infty)$, such that the following holds: If $y^{[2n]} = |r|f$ a.e. on $[0,b_m)$ with a function f such that $\int_0^{b_m} |r(t)||f(t)|^2 dt < \infty$, then

$$\int_0^{b_m} |r(t)||y(t)|^2 dt = 0 \quad \text{implies} \quad y(t) \equiv 0 \quad \text{on} \quad [0,b_m).$$

(Evidently, this implies that all the quasi-derivatives of y vanish on $[0,b_m)$ and $\int_0^{b_m} |r(t)||f(t)|^2 dt = 0$.)

REMARK 2.2. Assume that we have $y^{[2n]} = |r|f$ a.e. on $[0,b)$ with functions f, y, such that

$$\int_0^b |r(t)||f(t)|^2 dt < \infty, \quad \int_0^b |r(t)||y(t)|^2 dt = 0\,.$$

If assumption IV holds we find that y vanishes on each interval $[0,b_m)$, that is on $[0,b)$. Hence the assumption IV implies that its conclusion holds with b instead of b_m. In particular, for a given $y \in L^2_r$ the relation $y^{[2n]} = |r|f$ cannot hold with two different elements f of L^2_r.

The Condition IV serves a similar purpose as Condition B in Langer [28], p.15. In [6] conditions related to, but not identical with the basic assumptions I - IV are given (for the second-order case) under which the conclusions of the previous Remark are proved. It is possible to generalize these conditions to the $2n$-order case, and so to formulate conditions for I - IV to hold true. Instead of following this rather spare consuming scheme we have here preferred to use I - IV straight forwardly as a basis for the subsequent development.

2.3. Equivalent frist order system

From (2.1.1) it follows

$$\begin{aligned} &\frac{d}{dx}\, y^{[k-1]} = y^{[k]}, \quad k=1,2,\ldots,n-1 \\ &\frac{d}{dx}\, y^{[n-1]} = \frac{1}{p_0}\, y^{[n]} \\ &\frac{d}{dx}\, y^{[2n-k]} = p_{n-k+1}\, y^{[k-1]} - y^{[2n-k+1]}, \quad k=1,2,\ldots,n. \end{aligned} \tag{2.3.1}$$

The equation

$$y^{[2n]} = rf \quad \text{a.e. on} \quad [0,b), \tag{2.3.2}$$

where $rf \in L_{loc}[0,b)$, is equivalent to the first order system

$$K(\hat{y}(x))' = S(x)\hat{y}(x) + R(x)\hat{f}(x) \quad \text{on} \quad [0,b), \tag{2.3.3}$$

where $\hat{y}(x)$ and $\hat{f}(x)$ are the 2n-vectors:

$$\hat{y}(x) = (y(x),\, y^{[1]}(x),\ldots,\, y^{[2n-1]}(x))^T, \tag{2.3.4}$$

$$\hat{f}(x) = (f(x),\, 0,\ldots,0)^T \tag{2.3.5}$$

(T indicating the transpose), K is the $(2n\times 2n)$-matrix

$$K = \begin{pmatrix} 0 & -\tilde{I}_n \\ \tilde{I}_n & 0 \end{pmatrix} \quad \text{with} \quad \tilde{I}_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & & \iddots & 0 \\ \vdots & \iddots & & \vdots \\ 0 & \iddots & & \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

and $S(x)$ is the $(2n\times 2n)$-matrix

$$S(x) = \begin{bmatrix} -p_n & & & & & & 0 \\ & -p_{n-1} & & & & 0 & 1 \\ & & \ddots & & \iddots & \iddots & \\ & & & -p_1 & 0 & & \\ & & & 0 & \frac{1}{p_0} & & \\ & & \iddots & 1 & 0 & & \\ & 0 & \iddots & & & \ddots & \\ 0 & 1 & & & & & 0 \end{bmatrix} \leftarrow (n+1)\text{:th row},$$

$\uparrow$

(n+1):th column

where all elements other than indicated are zeros, and $R(x)$ is the $(2n\times 2n)$-matrix with elements

$$R_{ij}(x) = \begin{cases} r(x) & \text{if } i = j = 1 \\ 0 & \text{if } i,j = 1,2,\ldots,2n\,,\ i\cdot j \neq 1\,. \end{cases}$$

Evidently, K has the properties $K^2 = -I_{2n}$ (I_{2n} is the $(2n\times 2n)$-unit matrix), $K = -K^*$, and $S(x)$ is symmetric with elements (as functions of x) in $L_{loc}[0,b)$.

In this paper we are mainly interested in the equation $M(y) = r\,f$, where r changes sign, that is, there are subsets Δ_+ and Δ_- (in $[0,b)$) of positive Lebesgue measures such that

$$r(x) > 0 \text{ a.e. on } \Delta_+\,,\quad r(x) < 0 \text{ a.e. on } \Delta_-\,.$$

The study of this equation, however, will be based upon a treatment of

$$M(y) = |r|f \text{ on } [0,b)\,. \tag{2.3.6}$$

The first order system which is equivalent to (2.3.6) is

$$K(\hat{y}(x))' = S(x)\hat{y}(x) + |R|(x)\hat{f}(x)\,, \tag{2.3.7}$$

where $|R|(x)$ is the $(2n\times 2n)$-matrix with elements

$$(|R|(x))_{ij} = |R_{ij}(x)|\,,\ i,j=1,2,\ldots,2n\,.$$

2.4. Existence theorem (cf. Naimark [30], § 16.2)

If y is a solution of $y^{[2n]} = r\,f$ on $[0,b)$, then the vector function $\hat{y}$ in (2.3.4) is absolutely continuous on $[0,b)$, and it is an absolutely continuous solution of the first order system (2.3.3). Conversely, any such solution of the first order system has a first component $y(x) = y^{[0]}(x)$ which satisfies (2.3.2). Because of the properties of K, the first order system (2.3.3) can be written as

$$\hat{y}'(x) = -KS(x)\hat{y}(x) - r(x)\hat{f}(x)\,, \tag{2.4.1}$$

where $\hat{f}(x) = (0,\ 0,\ldots,0,\ f(x))^T$. The existence theorem for this system

is a special case of the one for the system (2.4.1) with $\hat{f}$ replaced by a more general vector-valued function

$$\hat{f}(x) = (f_{2n-1}, \ldots, f_2, f_1, f)^T$$

such that $\hat{f}$ is locally integrable on $[0,b)$.

This system is equivalent to the integral equation

$$\hat{y}(x) - \hat{y}_0 = -\int_0^x KS(t)\hat{y}(t)dt - \int_0^x r(t)\hat{f}(t)dt, \qquad (2.4.2)$$

which can be solved, e.g. by Picard's successive approximation technique. This gives a solution of (2.4.2) which is absolutely continuous on $[0,b)$. It is the unique solution of (2.4.1) which assumes an arbitrarily given vector value $\hat{y}_0$ for $x = 0$: $\hat{y}(0) = \hat{y}_0$.

Together with a solution $\hat{y}$ of (2.4.1), according to (2.3.4) we find a solution of the equation $y^{[2n]} = rf$ (or $y^{[2n]} = |r|f$) taking given initial values $y_0, \ldots, y_{2n-1}$:

$$y^{[k-1]}(0) = y_{k-1}, \; k = 1,2,\ldots,2n.$$

With $|R|(x)$ as in Section 2.3 we introduce for each $x \in [0,b)$ the corresponding hermitian form $\langle |R|(x)\xi,\eta\rangle$ $(\xi,\eta \in \mathbb{C}^{2n})$. Evidently,

$$\langle |R|(x)\eta,\eta\rangle \geq 0 \quad (\eta \in \mathbb{C}^{2n}).$$

The set of all measurable vector-functions

$$\hat{f}(x) = (f(x), f_1(x), \ldots, f_{2n-1}(x))^T, \qquad (2.4.3)$$

defined on $[0,b)$ with the property that the first component f belongs to L_r^2 is denoted by L. On L we define a (possibly degenerated) scalar product

$$(\hat{f},\hat{g})\hat{} = \int_0^b \langle |R|(x)\hat{f}(x), \hat{g}(x)\rangle \, dx. \qquad (2.4.4)$$

Put

$$L_0 = \{\hat{f} \neq 0: \hat{f} \in L, \; (\hat{f},\hat{f})\hat{} = 0\}.$$

Then the factor space $L_R^2 = L/L_0$ equipped with the scalar product (2.4.4)

is evidently a Hilbert space. The null element of L^2_R is the equivalence class of all measurable complex-valued 2n-vector-functions $\hat{f}$ on $[0,b)$ such that $(\hat{f},\hat{f})^{\wedge} = 0$.

The corresponding indefinite scalar product on L^2_R is defined by

$$[\hat{f},\hat{g}]^{\wedge} = \int_0^b \langle R(x)\hat{f}(x),\ \hat{g}(x)\rangle\, dx\,. \tag{2.4.5}$$

Now define a mapping $\mathscr{L}$ by

$$\mathscr{L}\colon L^2_R \longrightarrow L^2_r$$
$$\mathscr{L}\hat{f}(x) = f(x)\,,$$

if $\hat{f}$ and f are connected by (2.4.3), and from (2.4.4) it follows

$$(\mathscr{L}\hat{f}\,,\mathscr{L}\hat{g}) = (\hat{f},\hat{g})^{\wedge}\,.$$

That is, $\mathscr{L}$ is an isometric mapping from L^2_R onto L^2_r. In the following we shall write often $L^2_R[0,b)$ and $L^2_r[0,b)$ instead of L^2_R and L^2_r respectively.

2.5. Operators associated with $M(y) = |r|f$.

In [30], Naimark treats the equation $M(y) = f$ on an arbitrary real interval I. For this purpose he brings in the Hilbert space $L^2(I)$ and considers the set of functions y in $L^2(I)$ for which all quasi-derivatives of order $\leq 2n-1$ are AC_{loc} on I, and $y^{[2n]}$ belongs to $L^2(I)$. It is obvious that, for a given y in this set, there is at most one f in $L^2(I)$ for which $M(y) = f$. As a consequence, the relation $M(y) = f$ defines a linear operator L, $Ly = f$, in $L^2(I)$. Similarly, the equation $M(y) = |r|f$ can be treated in the Hilbert space $L^2_{|r|}(I)$ with the weight function $|r|$ provided $|r(x)| > 0$ on I. But if only $|r(x)| \geq 0$ is known to hold true, it may a priori be possible to have $y = 0$ (in $L^2_{|r|}(I)$), i.e. to have $\int_I |r(x)||y(x)|^2 dx = 0$, for a solution of $M(y) = |r|f$ even if f in $L^2_{|r|}(I)$ <u>is not</u> 0 <u>in this space</u>. As it was stated in Section 2.2 the basic assumption IV excludes this possibility so that $M(y) = |r|f$ defines a linear operator $\tilde{B}$, $\tilde{B}y = f$ in $L^2_{|r|}[0,b)$

with domain:

$$\tilde{D} = \{y \in L^2_{|r|}[0,b):\ y^{[2n]} \text{ exists},\ y^{[2n]} = |r|f \text{ a.e. in } [0,b) \text{ with } f \in L^2_{|r|}[0,b)\}.$$

In the sequel we put $L^2_r[0,b) = L^2_{|r|}[0,b)$.

Under the basic assumptions I - IV the endpoint b is called regular for the equation

$$M(y) = |r|f, \tag{2.5.1}$$

if $\int_0^b |r(x)|dx$, $\int_0^b \frac{1}{p_0(x)}dx$, $\int_0^b p_k(x)dx$, $k=1,2,\ldots,n$, are finite, otherwise b is called singular.

Corresponding to the considerations of Naimark [30] we introduce the restriction B'_0 of $\tilde{B}$ to the domain

$$D'_0 = \{y \in \tilde{D}:\ y(x) = 0 \text{ in some neighbourhood of } b,\ y^{[k]}(0) = 0,\ k=0,1,\ldots,2n-1\}.$$

We shall show, that D'_0 is a dense subset of $L^2_r[0,b)$ (c.f. [30], § 17).

For $y \in D'_0$ and $z \in \tilde{D}$ we have evidently from Lagrange's identity (2.1.2)

$$(B'_0 y,z) - (y,\tilde{B}z) = (M(y),z)_1 - (y,M(z))_1 = 0,$$

where $(\cdot,\cdot)$ is the scalar product (1.1.4) and $(\cdot,\cdot)_1$ denotes the scalar product in $L^2[0,b)$ (with respect to Lebesgue measure), and in particular

$$(B'_0 y,y) = (y,B'_0 y) \qquad (y \in D'_0),$$

that is B'_0 is symmetric.

Assume now that the endpoint b is regular for the equation (2.5.1) and let B_0 be the restriction of $\tilde{B}$ with domain:

$$D_0 = \{y \in \tilde{D}:\ y^{[k]}(0) = y^{[k]}(b) = 0,\ k=0,1,\ldots,2n-1\}.$$

Evidently

$$B'_0 \subset B_0 \subset \tilde{B}, \tag{2.5.2}$$

and from Lagrange's identity we have again

$$(B_0 y,z) = (y,\tilde{B}z) \quad (y \in D_0,\ z \in \tilde{D}),$$
$$(B_0 y,y) = (y,B_0 y) \quad (y \in D_0). \tag{2.5.3}$$

LEMMA 2.5.1 (cf. Naimark [30], § 17.3, Lemma 1). *Assume that the endpoint* b *is regular for the equation* (2.5.1) *and* $f \in L_r^2[0,b)$. *Then the equation* $M(y) = |r|f$ *has a solution* $y \in D_0$ *if and only if* $(f,z) = 0$ *for all solutions* z *of the homogeneous equation* $M(z) = 0$.

Proof. For given $f \in L_r^2[0,b)$ we have

$$\Big(\int_0^b |r(t)f(t)|dt\Big)^2 \le \int_0^b |r(t)|dt \int_0^b |r(t)||f(t)|^2 dt .$$

Therefore, by well-known existence theorems (compare Section 2.4), $M(y) = |r|f$ has a unique solution y on $[0,b]$ satisfying

$$y^{[k]}(0) = 0, \quad k = 0,1,\ldots,2n-1. \tag{2.5.4}$$

Denote by $z_1,\ldots,z_{2n}$ a fundamental system of solutions of the homogeneous equation $M(z) = 0$ satisfying

$$z_\nu^{[k-1]}(b) = \delta_{\nu k} \quad (\nu,k = 1,\ldots,2n).$$

These solutions are continuous on $[0,b]$, hence they belong to $L_r^2[0,b)$. Applying Lagrange's identity to the functions y and z_ν, we get

$$(f,z_\nu) = (M(y),z_\nu)_1 = h_b(y,z_\nu) - h_0(y,z_\nu) + (y,M(z_\nu))_1 .$$

But $M(z_\nu) = 0$ and, by the conditions (2.5.4), $h_0(y,z_\nu) = 0$. Hence the above relation gives

$$(f,z_\nu) = h_b(y,z_\nu) = \sum_{k=1}^{n} \left(y^{[k-1]}(b)\,\overline{z}_\nu^{[2n-k]}(b) - y^{[2n-k]}(b)\,\overline{z}_\nu^{[k-1]}(b)\right)$$
$$= \begin{cases} -y^{[2n-\nu]}(b) & \text{for } \nu = 1,2,\ldots,n, \\ y^{[2n-\nu]}(b) & \text{for } \nu = (n+1),\ldots,2n, \end{cases}$$

the equations $y^{[k]}(b) = 0$, $k = 0,1,\ldots,(2n-1)$ are satisfied if and only if

$$(f,\, z_\nu) = 0 \quad \text{for } \nu = 1,\ldots,2n.$$

Denote by $\mathcal{M}$ the solutions of the equation $M(z) = 0$. As all these solutions are continuous on $[0,b]$, they belong to $L_r^2[0,b)$: $\mathcal{M} \subset L_r^2[0,b)$. The dimension of $\mathcal{M}$ as a subspace of $L_r^2[0,b)$ is also $2n$: Otherwise, for some $y \neq 0$ we would have $y = 0$ in $L_r^2[0,b)$, $M(y) = 0$, a contradiction to our assumption IV.

Now, as in Naimark [30], § 17.3, if $\mathcal{R}_0$ denotes the range of B_0, we have from Lemma 2.5.1 with $L_r^2 = L_r^2[0,b)$:

$$L_r^2 = \mathcal{R}_0 \oplus \mathcal{M}. \tag{2.5.5}$$

Also the proof of the following lemma can be taken from Naimark [30], § 17.3, Lemma 2.

LEMMA 2.5.2. *For arbitrary numbers* $\alpha_0,\dots,\alpha_{2n-1}, \beta_0,\dots,\beta_{2n-1}$ *there is a function* y *belonging to* $\tilde{D}$ *such that*

$$y^{[k]}(0) = \alpha_k, \quad y^{[k]}(b) = \beta_k, \quad k = 0,1,\dots,2n-1.$$

PROPOSITION 2.5.3. *If the endpoint* b *is regular for the equation* (2.5.1), *then the domain of definition* D_0 *of the operator* B_0 *is dense in* L_r^2.

We have to show that every element h in L_r^2 which is orthogonal to D_0, is the zero element. Assume $(h,y) = 0$ for all $y \in D_0$. Let z be any solution of the equation $M(z) = |r|h$: $\tilde{B}z = h$. By (2.5.3) for an arbitrary element y in D_0, we have

$$(z, By) = (\tilde{B}z, y) = (h,y) = 0$$

that is z is orthogonal to $\mathcal{R}_0$. It follows from (2.5.5) that z belongs to $\mathcal{M}$, hence $M(z) = 0$, or $|r|h = 0$ a.e. on $[0,b)$, that is $h = 0$ in L_r^2.

PROPOSITION 2.5.4. *If the endpoint* b *is singular for the equation* (2.5.1), *then the domain of definition* D_0' *of the operator* B_0' *is dense in* L_r^2.

We have to show that for every element h in L_r^2 which is orthogonal

to D_0' we have $h = 0$ in L_r^2. Choose a b_m according to assumption IV and let $\Delta = [0,b_m] \subset [0,b)$. Then, every function y in $D_{0,\Delta}$, where

$$D_{0,\Delta} = \{y \in L_r^2[0,b_m): y^{[2n]} \text{ exists on } [0,b_m],\ y^{[2n]} = |r|f \text{ a.e. in } [0,b_m) \text{ with } f \in L_r^2[0,b_m),\ y^{[k]}(0) = y^{[k]}(b_m) = 0,\ k = 0,1,\dots, 2n-1\}$$

can be regarded as an element of D_0' (putting it zero on $[b_m,b)$), hence h is orthogonal to $D_{0,\Delta}$ in $L_r^2[0,b_m)$. By Proposition 2.5.3, $D_{0,\Delta}$ is dense in $L_r^2[0,b_m)$ and it follows $r(x)h(x) = 0$ a.e. on $[0,b_m)$. As m is arbitrary we get $r(x)h(x) = 0$ a.e. on $[0,b)$ that is $h = 0$ in $L_r^2[0,b)$.

Since the operator B_0' is symmetric, it admits a closure which is denoted by B.

PROPOSITION 2.5.5. *The adjoint of* B *is* $\tilde{B}$, *that is* $B^* = \tilde{B}$.

After the statements 2.5.1 - 2.5.4 being proved, this proposition follows in the same way as Proposition I in [30], § 17.4.

The structure of the operator B depends upon the character of the endpoint b. The pair of deficiency indices of B has the form (m,m) since the differential expression M and the function r are real, and $n \le m \le 2n$, one endpoint being regular ([30]). In addition to the basic assumptions I - IV we often suppose *that* b *is singular and that the deficiency index of the operator* B *is equal to* n. In the vocabulary of [10] this means that $M(y) = |r|f$ is in limit (n,n) case at b. In case $n = 1$ it means that the equation (2.5.1) is in Weyl's limit point case at b.

2.6. Generalization of Sturm-Liouville's boundary condition

In this section we suppose, that b is singular and (2.5.1) is in limit (n,n) case at b. We are interested in selfadjoint extensions B_α of B

(the notation will be motivated later on). According to Proposition 2.5.5, $B \subset B_\alpha$ implies $B_\alpha \subset \tilde{B}$, that is, B_α is a selfadjoint restriction of $\tilde{B}$. Since (2.5.1) is in limit (n,n) case at b, we have (see Pleijel [37])

$$h_b(u,v) = \lim_{x \to b} h_x(u,v) = 0 \quad (u,v \in \tilde{D}) . \tag{2.6.1}$$

To every y in $\tilde{D}$ there is a vector $\hat{y}_0 = (y(0),\dots,y^{[2n-1]}(0))^T$ in $\mathbb{C}^{2n}$, and vice versa by Lemma 2.5.2. In the following we put $\hat{y} = \hat{y}_0$; no confusion shall be made with the vector function $\hat{y}$ in (2.3.4). Under the mapping $y \to \hat{y}$ we have

$$h_0(u,v) = \langle K\hat{u},\hat{v}\rangle , \tag{2.6.2}$$

where K is the $(2n \times 2n)$-matrix of Section 2.3.

Consider the image H_α in $\mathbb{C}^{2n}$ of the mapping $y \to \hat{y}$, $y \in \tilde{D}$. The condition for B_α to be a selfadjoint restriction of $\tilde{B}$ means (comp. [30], § 18.1, Theorem 1) that H_α is a maximal neutral subspace with respect to the scalar product (2.6.2), that is

(1) $\langle K\hat{u},\hat{v}\rangle = 0$ if $\hat{u},\hat{v} \in H_\alpha$;

(2) if $\langle K\hat{u},\hat{u}\rangle = 0$ and $\langle K\hat{u},\hat{v}\rangle = 0$ for all $\hat{v} \in H_\alpha$ then $\hat{u} \in H_\alpha$.

Because of the special structure of K, instead of (2) we can write

(2′) $\dim H_\alpha = n$.

The form (2.6.2) is not hermitian, but $i\langle K\hat{u},\hat{v}\rangle$ $(\hat{u},\hat{v} \in \mathbb{C}^{2n})$ is. Put

$$P_+ = \tfrac{1}{2}(I_{2n} + iK) , \quad P_- = \tfrac{1}{2}(I_{2n} - iK) .$$

From the properties of K, $K^2 = -I_{2n}$, $K^* = -K$, it follows that P_+ and P_- are orthogonal projectors in $\mathbb{C}^{2n}$. Clearly,

$$P_+ + P_- = I_{2n} , \quad iK = P_+ - P_- .$$

Since H_α is linear, condition (1) means that

$$\langle iK\hat{u},\hat{u}\rangle = 0 \quad \text{for all} \quad \hat{u} \in H_\alpha ,$$

and this relation holds if and only if

$$\|P_+\hat{u}\| = \|P_-\hat{u}\| \quad \text{for all} \quad \hat{u} \in H_\alpha ; \tag{2.6.3}$$

here $\|\hat{u}\| = \langle \hat{u}, \hat{u} \rangle^{1/2}$ $(u \in \mathbb{C}^{2n})$.

Relation (2.6.3) implies that there is a unitary mapping V_α from $P_+ H_\alpha$ onto $P_- H_\alpha$ and such that $V_\alpha P_+ \hat{u} = P_- \hat{u}$ if $\hat{u} \in H_\alpha$. Conversely, given a unitary mapping V_α from $P_+ H_\alpha$ onto $P_- H_\alpha$, then the set $\{P_+ \hat{u} + V_\alpha P_- \hat{u}: \hat{u} \in \mathbb{C}^{2n}\}$ is a maximal neutral subspace of $\mathbb{C}^{2n}$.

If $\hat{y} \in \mathbb{C}^{2n}$, $\hat{y} = (\hat{y}_1, \hat{y}_2)^T$ with $\hat{y}_1, \hat{y}_2 \in \mathbb{C}^n$, the definition of $P_\pm$ implies

$$P_+ \hat{y} = \frac{1}{2} \begin{pmatrix} \hat{y}_1 - i\tilde{I}_n \hat{y}_2 \\ \hat{y}_2 + i\tilde{I}_n \hat{y}_1 \end{pmatrix}, \quad P_- \hat{y} = \frac{1}{2} \begin{pmatrix} \hat{y}_1 + i\tilde{I}_n \hat{y}_2 \\ \hat{y}_2 - i\tilde{I}_n \hat{y}_1 \end{pmatrix}.$$

Here, evidently, the second components are obtained from the first components by applying the mapping $i\tilde{I}_n$ or $-i\tilde{I}_n$ respectively. It follows that the mapping V_α can be represented by an $(n \times n)$-matrix, mapping the first component of $P_+ \hat{y}$ isometrically onto the first component of $P_- \hat{y}$. Writing this unitary $(n \times n)$-matrix as $-\exp(-2i\alpha)$ with some symmetric $(n \times n)$-matrix α, whose eigenvalues are in the interval $[0, \pi)$, the condition (2.6.3) takes the form

$$e^{-i\alpha}(\hat{y}_1 - i\tilde{I}_n \hat{y}_2) + e^{i\alpha}(\hat{y}_1 + i\tilde{I}_n \hat{y}_2) = 0. \tag{2.6.4}$$

If we introduce the $(n \times n)$-matrices $\cos\alpha$ and $\sin\alpha$, then (2.6.4) can be written as

$$(\cos\alpha)\,\hat{y}_1 - (\sin\alpha)\,\tilde{I}_n \hat{y}_2 = 0.$$

Here (as sometimes in the following) we have written $(\cos\alpha)$ and $(\sin\alpha)$ instead of $\cos\alpha$ and $\sin\alpha$, emphasizing that these quantities are $(n \times n)$-matrices.

Moreover, as we always want to deal with real operators, we suppose additionally that the matrix α is real. Summing up we have the following

PROPOSITION 2.6.1. Suppose that the equation (2.5.1) is in limit (n,n) case at b. Then every real selfadjoint restriction B_α of the operator $\tilde{B}$ (extension of the operator B) is determined by a boundary condition

$$(\cos\alpha)\,(y(0),\ y^{[1]}(0),\dots,y^{[n-1]}(0))^T - $$
$$- (\sin\alpha)\,\tilde{I}_n(y^{[n]}(0),\dots,y^{[2n-1]}(0))^T = 0\,, \tag{2.6.5}$$

<u>with a real symmetric</u> $(n\times n)$-<u>matrix</u> α <u>whose eigenvalues are in the interval</u> $[0,\pi)$. <u>Conversely, for every such matrix</u> α <u>the boundary condition</u> (2.6.5) <u>determines a real selfadjoint restriction</u> B_α <u>of</u> $\tilde{B}$.

In the following we shall consider the differential equation (2.5.1) also when it is not in limit (n,n) case at b. For this purpose we introduce also for a given matrix α with the above mentioned properties the set

$$D'_\alpha = \{y \in \tilde{D}:\ y \equiv 0 \text{ in a neighbourhood of } b,\ (2.6.5) \text{ holds}\}\,.$$

Let B'_α be the restriction of $\tilde{B}$ to D'_α. Then B'_α is symmetric. Indeed, the Lagrangean identity integrated over $[0,b)$ with $u,v \in D'_\alpha$ gives

$$(B'_\alpha u,v) - (u,B'_\alpha v) = -h_0(u,v) = -\langle K\hat{u},\hat{v}\rangle = 0\,,$$

where the last equality sign follows from (1)(see p.2.6.2). Since B'_α is symmetric, it has a closure which is denoted by B_α. Of course, in this more general case B_α is only symmetric, not necessarily selfadjoint.

If B is regular and β is also a real symmetric $(n\times n)$-matrix with eigenvalues in $[0,\pi)$, it follows by a similar reasoning as above that the restriction of $\tilde{B}$ to the domain

$$\{y \in \tilde{D}:\ (\cos\alpha)\,\hat{y}_1(0) - (\sin\alpha)\,\tilde{I}_n\hat{y}_2(0) = 0\,,$$
$$(\cos\beta)\,\hat{y}_1(0) + (\sin\beta)\,\tilde{I}_n\hat{y}_2(0) = 0\} \tag{2.6.6}$$

is a real selfadjoint operator.

The boundary conditions (2.6.5), (2.6.6) can be considered as generalizations of the classical Sturm-Liouville boundary conditions.

2.7. Generalization of Weyl's function

In this section we shall consider the equation

$$y^{[2n]} = \lambda|r|y \quad \text{on} \quad [0,b) \tag{2.7.1}$$

and the corresponding first order system

$$K\hat{y}' = (S + \lambda|R|)\,\hat{y} \quad \text{on} \quad [0,b) \tag{2.7.2}$$

(see Section 2.3). Similar to H. Weyl's treatment of the second-order case with limit-point at b, for arbitrary complex λ we consider solutions $\varphi_k(\cdot,\lambda)$, $\theta_k(\cdot,\lambda)$, $k = 0,1,\ldots,n-1$ (which depend on α), of the equation (2.7.1) such that the matrix function

$$U(\cdot,\lambda) = \left(\begin{array}{ccc:ccc} \varphi_0(\cdot,\lambda) & \cdots\cdots & \varphi_{n-1}(\cdot,\lambda) & \theta_0(\cdot,\lambda) & \cdots\cdots & \theta_{n-1}(\cdot,\lambda) \\ \varphi_0^{[1]}(\cdot,\lambda) & \cdots & \varphi_{n-1}^{[1]}(\cdot,\lambda) & \theta_0^{[1]}(\cdot,\lambda) & \cdots & \theta_{n-1}^{[1]}(\cdot,\lambda) \\ \vdots & & & \vdots & & \\ \varphi_0^{[n-1]}(\cdot,\lambda) & \cdots & \varphi_{n-1}^{[n-1]}(\cdot,\lambda) & \theta_0^{[n-1]}(\cdot,\lambda) & \cdots & \theta_{n-1}^{[n-1]}(\cdot,\lambda) \\ \hdashline \varphi_0^{[n]}(\cdot,\lambda) & \cdots & \varphi_{n-1}^{[n]}(\cdot,\lambda) & \theta_0^{[n]}(\cdot,\lambda) & \cdots & \theta_{n-1}^{[n]}(\cdot,\lambda) \\ \vdots & & & \vdots & & \\ \varphi_0^{[2n-1]}(\cdot,\lambda) & \cdots & \varphi_{n-1}^{[2n-1]}(\cdot,\lambda) & \theta_0^{[2n-1]}(\cdot,\lambda) & \cdots & \theta_{n-1}^{[2n-1]}(\cdot,\lambda) \end{array}\right)$$

or, with evident notation

$$U(\cdot,\lambda) = \left(\begin{array}{c:c} \varphi(\cdot,\lambda;1) & \theta(\cdot,\lambda;1) \\ \hdashline \varphi(\cdot,\lambda;2) & \theta(\cdot,\lambda;2) \end{array}\right) = \left(\varphi(\cdot,\lambda) \;\vdots\; \theta(\cdot,\lambda)\right) \tag{2.7.3}$$

satisfies the following boundary condition at $x = 0$:

$$U(0,\lambda) = \begin{pmatrix} \varphi(0,\lambda;1) & \theta(0,\lambda;1) \\ \varphi(0,\lambda;2) & \theta(0,\lambda;2) \end{pmatrix} = \begin{pmatrix} (\sin\alpha) & (\cos\alpha) \\ \tilde{I}_n(\cos\alpha) & -\tilde{I}_n(\sin\alpha) \end{pmatrix}. \tag{2.7.4}$$

It is easy to see that $U(\cdot,\lambda)$ is a solution of the matrix differential equation

$$K\frac{d}{dx}U(x,\lambda) = S(x)U(x,\lambda) + \lambda|R|(x)U(x,\lambda) \quad (x \in [0,b), \text{ a.e.})$$

corresponding to (2.7.2).

With

$$(K\frac{d}{dx}(U(x,\mu))^* = ((S(x)+\mu|R|(x))U(x,\mu))^* \quad (x \in [0,b))$$

a simple calculation gives

$$\frac{d}{dx}(U(x,\mu)^* K U(x,\lambda)) = (\lambda - \overline{\mu})U(x,\mu)^* |R|(x)U(x,\lambda) . \tag{2.7.5}$$

For $\mu = \overline{\lambda}$ it follows that $U(x,\overline{\lambda})^* K U(x,\lambda)$ is independent of x and its value can be found by the help of (2.7.4):

$$U(x,\overline{\lambda})^* K U(x,\lambda) = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{2.7.6}$$

from which it follows that

$$|\det U(x,\lambda)| = 1 \quad (x \in [0,b) , \ \lambda \in \mathbb{C}) .$$

This implies that the functions $\varphi_0,\dots,\varphi_{n-1}$, $\theta_0,\dots,\theta_{n-1}$ are linearly independent. The matrix function $U(\cdot,\lambda)$ is the so-called fundamental matrix solution of (2.7.2), (2.7.4); sometimes we shall denote by $U_\alpha(\cdot,\lambda)$, in order to emphasize its dependence on α. If β is a matrix with the same properties as α, then we have

$$U_\beta(x;\lambda) = U_\alpha(x;\lambda) C \tag{2.7.7}$$

with a constant non-singular matrix C, which can be found from the initial conditions (2.7.4) as

$$C = \begin{pmatrix} (\sin\alpha)(\cos\beta)+(\cos\alpha)(\cos\beta) & (\sin\alpha)(\cos\beta)-(\cos\alpha)(\sin\beta) \\ (\cos\alpha)(\cos\beta)-(\sin\alpha)(\cos\beta) & (\sin\alpha)(\sin\beta)+(\cos\alpha)(\cos\beta) \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} . \tag{2.7.8}$$

It is now assumed that $M(y) = \lambda|r|y$ is in limit (n,n) case at the endpoint b (while 0 is regular). This means that the hermitian form $ih_b(u,u) = 0$ $(u,v \in \tilde{D})$ by (2.6.1), which according to a theorem by K. Kodaira (in generalized version, see Pleijel [37]) is equivalent to the condition that

$$i\int_0^b (M(u)\overline{v} - u\overline{M(v)})dx$$

has the signature (n,n) on the space of solutions of $M(y) = \lambda|r|y$ for any non-real value of λ. Because of $M(y) = \lambda|r|y$

$$i(\lambda - \bar{\lambda}) \int_0^b |r| u \bar{v}\, dx$$

has the same signature (n,n) on the same linear space. This is finally equivalent to the condition that $M(u) = \lambda|r|u$ <u>for any non-real</u> λ <u>has exactly</u> n <u>linearly independent solutions in the Hilbert space</u> $L_r^2[0,b)$, i.e. <u>such that</u>

$$\int_0^b |r||u|^2 dx < \infty,$$

see Pleijel [37], [32].

The solutions $\hat{u}$ of (2.7.2) for which the corresponding u belongs to $L_r^2[0,b)$ span an n-dimensional space which is momentarily denoted by S_n. A base in S_n is the set of columns in

$$U_\alpha V_n,$$

where V_n consists of n linearly independent columns

$$v_j = \begin{pmatrix} v_j(1) \\ v_j(2) \end{pmatrix}, \quad j = 0,1,\ldots,n-1,$$

with $v_j(1) = (v_{0j},\ldots,v_{n-1\,j})^T$ and $v_j(2) = (v_{nj},\ldots,v_{2n-1\,j})^T$, so that

$$V_n = \begin{pmatrix} V_n(1) \\ V_n(2) \end{pmatrix}.$$

Since V_n consists of linearly independent columns it follows that $V_n(1)$ and $V_n(2)$ enjoy the same property. This can be seen, e.g. for $V_n(2)$, as follows.

Assume that the vectors $v_j(2)$, $j = 0,1,\ldots,n-1$ are not linearly independent. Then we have, for some constants $c_0,\ldots,c_{n-1}$, $\sum_{j=0}^{n-1} |c_j| \neq 0$:

$$\sum_{j=0}^{n-1} c_j v_j(2) = 0,$$

which implies

$$U_\alpha(x,\lambda) \sum_{j=0}^{n-1} c_j v_j = U_\alpha(x,\lambda)\, w,$$

where $w = (w_0,\dots,w_{n-1}, 0,\dots,0)^T$ belongs to $\mathbb{C}^{2n}$. The first component of the right hand side belongs to L^2_r and it can be written as

$$\sum_{k=0}^{n-1} w_k\varphi_k(x,\lambda) = f(x,\lambda) \in L^2_r .$$

Moreover, we have

$$(\cos\alpha)\,\hat{f}_1(0,\lambda) - (\sin\alpha)\,\tilde{I}_n\hat{f}_2(0,\lambda) = 0 ,$$

which means that $\lambda \in \sigma_p(B_\alpha)$. But this is impossible for nonreal λ.

All other bases in S_n are obtained as

$$U_\alpha V_n D = U_\alpha \begin{pmatrix} V_n(1)\,D \\ V_n(2)\,D \end{pmatrix},$$

where D is a non-singular $(n\times n)$-matrix. By the choice $D = (V_n(2))^{-1}$ one obtains the base

$$U_\alpha \begin{pmatrix} M \\ I_n \end{pmatrix}$$

of S_n. It is readily seen that $M = V_n(1)(V_n(2))^{-1}$ is independent of the choice of V_n giving linearly independent solutions of (2.7.2). In accordance with the notations in Titchmarsh's presentation of Weyl's theory in [39] we write

$$U_\alpha \begin{pmatrix} M \\ I_n \end{pmatrix} = (\psi_0, \psi_1,\dots,\psi_{n-1}) .$$

If $M = (m_{jk})_{j,k=0}^{n-1}$ (not to be confused with the differential expression M) this explicitly reads

$$\psi_j^{[k]}(x,\lambda) = \theta_j^{[k]}(x,\lambda) + \sum_{\ell=0}^{n-1} \varphi^{[k]}(x,\lambda)m_{\ell j}(\lambda) ,$$

$j = 0,1,\dots,n-1$, $0 \le k \le 2n-1$ and $\psi_j(\cdot,\lambda) \in L^2_r[0,b)$ for $j = 0,1,\dots,n-1$. Compare Titchmarsh [39].

The dependence of the matrix M on α and λ is observed by writing $M_\alpha(\lambda)$ $(= (m_{jk,\alpha}(\lambda))_{j,k=0}^{n-1})$ instead of M. However, in the following the index α shall often be omitted. Now the above relation can be written as

$$\begin{pmatrix} \psi(x,\lambda;1) \\ \psi(x,\lambda;2) \end{pmatrix} = \begin{pmatrix} \varphi(x,\lambda;1) & \theta(x,\lambda;1) \\ \varphi(x,\lambda;2) & \theta(x,\lambda;2) \end{pmatrix} \begin{pmatrix} M(\lambda) \\ I_n \end{pmatrix} . \qquad (2.7.9)$$

Since the initial conditions (2.7.5) are real and independent of λ, it follows that

$$m_{jk}(\bar{\lambda}) = \overline{m_{kj}(\lambda)} \qquad (j,k = 0,1,\ldots,n-1)$$

$$M_\alpha(\bar{\lambda}) = M_\alpha(\lambda)^*.$$

For another matrix β the columns of $U_\beta \begin{pmatrix} M_\beta \\ I_n \end{pmatrix}$ span the same space S_n as the columns of $U_\alpha \begin{pmatrix} M_\alpha \\ I_n \end{pmatrix}$. Therefore

$$U_\alpha \begin{pmatrix} M_\alpha \\ I_n \end{pmatrix} \hat{u} = U_\beta \begin{pmatrix} M_\beta \\ I_n \end{pmatrix} \hat{v}$$

with two n-vectors $\hat{u}$ and $\hat{v}$ of which one can be taken arbitrary.

Using (2.7.7), it follows

$$\begin{pmatrix} M_\alpha \\ I_n \end{pmatrix} \hat{u} = C \begin{pmatrix} M_\beta \\ I_n \end{pmatrix} \hat{v} ,$$

and an easy computation gives the relation

$$M_\alpha(\lambda) = (C_{11}M_\beta + C_{12})(C_{21}M_\beta + C_{22})^{-1} \tag{2.7.10}$$

with $(n\times n)$-matrices C_{ij} from (2.7.8), $i,j = 1,2$.

For $\alpha = 0$ (nullmatrix), $\beta = \frac{\pi}{2} I_n$ one obtains

$$M_0(\lambda) = - M^{-1}_{\frac{\pi}{2}I_n}(\lambda) . \tag{2.7.11}$$

The following lemma is important in the sequel.

LEMMA 2.7.1. <u>The function</u>

$$\lambda \longrightarrow M_{\frac{\pi}{2}I_n}(\lambda)$$

<u>belongs to the class</u> $N_0(\mathbb{C}^n)$.

<u>Proof</u>. By Theorem I in W.J. Donoghue, Jr [8], p.95, it is enough to show that the matrix

$$\frac{M_{\frac{\pi}{2}I_n}(\lambda) - M_{\frac{\pi}{2}I_n}(\bar{\lambda})}{\lambda - \bar{\lambda}} \qquad (\lambda \neq \bar{\lambda})$$

is a nonnegative matrix.

From the relations

$$M(\psi_j(\cdot,\lambda)) = \lambda |r| \psi_j(\cdot,\lambda), \quad j = 0,1,\dots,n-1$$

and the Lagrangean identity one gets for $\lambda \neq \bar{\mu}$

$$(\lambda - \bar{\mu}) \int_0^b |r(x)|\, \psi_j(x,\lambda)\, \overline{\psi_k(x,\mu)}\,dx = -h_0(\psi_j(x,\lambda), \psi_k(x,\mu)) =$$

$$= -\langle K\hat{\psi}_j(0,\lambda), \hat{\psi}_k(0,\mu)\rangle =$$

$$= \langle \tilde{I}_n \hat{\psi}_j(0,\lambda;2), \hat{\psi}_k(0,\mu;1)\rangle - \langle \tilde{I}_n \hat{\psi}_j(0,\lambda;1), \hat{\psi}_k(0,\mu;2)\rangle ,$$

where $\hat{\psi}_j(0,\lambda;\ell)$ stands for the ℓ:th column of the matrix $\psi(0,\lambda;\ell)$, $\ell = 1,2$, see (2.7.9).

The right hand side of the last relation is

$$\langle \tilde{I}_n (\theta_j(0,\lambda;2) + (\varphi(0,\lambda;2)M_\alpha(\lambda))_j),\ \theta_k(0,\mu;1) + (\varphi(0,\mu;1)M_\alpha(\mu))_k\rangle -$$

$$- \langle \tilde{I}_n (\theta_j(0,\lambda;1) + (\varphi(0,\lambda;1)M_\alpha(\lambda))_j),\ \theta_k(0,\mu;2) + (\varphi(0,\mu;2)M_\alpha(\mu))_k\rangle .$$

By (2.7.5) this equals

$$\langle -(\sin\alpha)_j + ((\cos\alpha)\, M_\alpha(\lambda))_j,\ (\cos\alpha)_k + ((\sin\alpha)\, M_\alpha(\mu))_k\rangle +$$

$$+ \langle (\cos\alpha)_j + (\sin\alpha\, M_\alpha(\lambda))_j,\ (\sin\alpha)_k + (\cos\alpha\, M_\alpha(\mu))_k\rangle .$$

For $\alpha = \frac{\pi}{2} I_n$, one gets with $m_{jk}(\lambda) = m_{jk;\pi/2}(\lambda)$

$$(\lambda - \bar{\mu}) \int_0^b |r(x)|\, \psi_j(x,\lambda)\, \overline{\psi_k(x,\mu)}\,dx = m_{jk}(\lambda) - m_{kj}(\bar{\mu})$$

or

$$(\psi_j(\cdot,\lambda), \psi_k(\cdot,\mu)) = \frac{m_{jk}(\lambda) - m_{kj}(\bar{\mu})}{\lambda - \bar{\mu}} ,$$

where $(\cdot,\cdot)$ is the scalar product in the Hilbert space $L^2_r[0,b)$. Hence

$$(\psi_j(\cdot,\lambda), \psi_k(\cdot,\mu))_{j,k=0}^{n-1} = \frac{M_{\frac{\pi}{2}I_n}(\lambda) - M_{\frac{\pi}{2}I_n}(\bar{\mu})}{\lambda - \bar{\mu}} .$$

Now the statement follows, because the left hand side of this relation is a positive matrix if $\mu = \lambda$.

THEOREM 2.7.2. The function

$$\lambda \longrightarrow M_\alpha(\lambda) \quad (0 \leq \alpha < \pi\, I_n)$$

belongs to the class $N_0(\mathbb{C}^n)$.

Proof. With $\beta = \frac{\pi}{2} I_n$ one gets from (2.7.10)

$$M_\alpha(\lambda) = \left[(\sin\alpha)\, M_{\frac{\pi}{2}I_n}(\lambda) - (\cos\alpha)\right]\left[(\sin\alpha) + (\cos\alpha)\, M_{\frac{\pi}{2}I_n}(\lambda)\right]^{-1} .$$

$$\frac{M_\alpha(\lambda) - M_\alpha(\mu)^*}{\lambda - \bar{\mu}} = \frac{1}{\lambda - \bar{\mu}} \Big\{ \left[(\sin\alpha)\, M_{\frac{\pi}{2}I_n}(\lambda) - (\cos\alpha)\right]\left[(\sin\alpha) + (\cos\alpha)\, M_{\frac{\pi}{2}I_n}(\lambda)\right]^{-1} -$$

$$- \left[(\sin\alpha) + M_{\frac{\pi}{2}I_n}(\mu)^*(\cos\alpha)\right]^{-1}\left[M_{\frac{\pi}{2}I_n}(\mu)^*(\sin\alpha) - (\cos\alpha)\right] =$$

$$= \frac{1}{\lambda - \bar{\mu}} \left[(\sin\alpha) + M_{\frac{\pi}{2}I_n}(\mu)^*(\cos\alpha)\right]^{-1} \Big\{ \left[(\sin\alpha) + M_{\frac{\pi}{2}I_n}(\mu)^*(\cos\alpha)\right]\left[(\sin\alpha)\, M_{\frac{\pi}{2}I_n}(\lambda) - (\cos\alpha)\right] -$$

$$- \left[M_{\frac{\pi}{2}I_n}(\mu)^*(\sin\alpha) - (\cos\alpha)\right]\left[(\sin\alpha) + (\cos\alpha)\, M_{\frac{\pi}{2}I_n}(\lambda)\right] \Big\} \left[(\sin\alpha) + (\cos\alpha)\, M_{\frac{\pi}{2}I_n}(\lambda)\right]^{-1} =$$

$$= \left[(\sin\alpha) + M_{\frac{\pi}{2}I_n}(\mu)^*(\cos\alpha)\right]^{-1} \left\{ \frac{M_{\frac{\pi}{2}I_n}(\lambda) - M_{\frac{\pi}{2}I_n}(\mu)^*}{\lambda - \bar{\mu}} \right\} \left[\sin\alpha) + (\cos\alpha)\, M_{\frac{\pi}{2}I_n}(\lambda)\right]^{-1}$$

and from Lemma 2.7.1 the statement follows.

It is worth while to remarque that according to (2.7.5) with $\psi = \varphi M + \theta$ one gets

$$\frac{d}{dx}\left(\psi(x,\mu)^* K \psi(x,\lambda)\right) = (\lambda - \bar{\mu})\, \psi(x,\mu)^*\, |R|(x)\, \psi(x,\lambda) .$$

Integrating this relation

$$\psi(0,\mu)^* K \psi(0,\lambda) = (\lambda - \bar{\mu}) \int_0^b \psi(x,\mu)^* |R|(x) \psi(x,\lambda)\, dx .$$

According to (2.7.9) $\psi = U \begin{pmatrix} M \\ I_n \end{pmatrix}$, and due to (2.7.6′)

$$\psi(0,\mu)^* K \psi(0,\lambda) = (M(\mu)^*\, I_n)\; U(0,\mu)^* K U(0,\lambda) \begin{pmatrix} M(\lambda) \\ I_n \end{pmatrix} =$$

$$= (M(\mu)^* \quad I_n) \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} M(\lambda) \\ I_n \end{pmatrix} = M(\mu)^* - M(\lambda) .$$

Hence, one obtains the formula (corresponding to a relation in Titchmarsh [39])

$$\frac{M(\lambda) - M(\mu)^*}{\lambda - \bar{\mu}} = \int_0^b \psi(x,\mu)^* |R|(x)\psi(x,\lambda)dx \qquad (2.7.12)$$

2.8. Integral representation of the resolvent .

In the following $|R|(x)$, $S(x)$ and K denote the $(2n\times 2n)$-matrices as given in § 2.3. We also consider the spaces L'_r and L'_R defined as follows. L'_r is the set of all f belonging to L^2_r such that f vanishes near b. That is there exists a $b' < b$, depending on f, such that $f(x) = 0$ for $b' \le x < b$. L'_R is the set of all vector valued functions $\hat{f}(x)$ in L^2_R whose components vanish in the same way near b.

For a given $f \in L'_r$, we shall construct a solution g of the inhomogeneous equation, λ a complex number,

$$y^{[2n]} - \lambda|r|y = |r|f \quad \text{on} \quad [0,b) \qquad (2.8.1)$$

satisfying the initial condition at zero

$$(\cos\alpha)\hat{y}_1(0) - (\sin\alpha)\tilde{I}_n\hat{y}_2(0) = 0 . \qquad (2.8.2)$$

If the matrix $M_\alpha(\lambda) = M(\lambda)$ of Section 2.7 is non-singular, the unique solution will be found in the form

$$g(x,\lambda) = \sum_{k=0}^{n-1} \psi_k(x,\lambda) \int_0^x |r(t)|f(t)\varphi_k(t,\lambda)dt + \varphi_k(x,\lambda) \int_x^b |r(t)|f(t)\psi_k(t,\lambda)dt, \qquad (2.8.3)$$

where φ_k, ψ_k are the functions in Section 2.7.

Consider the equivalent first order system of (2.8.1), i.e.

$$K\hat{y}'(x) - (S(x) - \lambda|R|(x)\hat{y}(x) = |R|(x)\hat{f}(x) \qquad (2.8.4)$$

under the initial condition (2.8.2) which is satisfied if

$$(\hat{y}_1(0), \hat{y}_2(0))^T = ((\sin\alpha)\hat{c}\,, \tilde{I}_n(\cos\alpha)\hat{c})^T \qquad (2.8.5)$$

with an arbitrary constant n-vector $\hat{c}$.

Under the basic assumptions I - IV, the initial-value problem (2.8.4 - 5) has a unique solution. The matrix function

$$V(x,\lambda) = \begin{pmatrix} \psi(x,\lambda;1) & \varphi(x,\lambda;1) \\ \psi(x,\lambda;2) & \varphi(x,\lambda;2) \end{pmatrix} = \begin{pmatrix} \varphi(x,\lambda;1) & \theta(x,\lambda;1) \\ \varphi(x,\lambda;2) & \theta(x,\lambda;2) \end{pmatrix} \begin{pmatrix} M(\lambda) & I_n \\ I_n & 0 \end{pmatrix} =$$

$$= U(x,\lambda) \begin{pmatrix} M(\lambda) & I_n \\ I_n & 0 \end{pmatrix}$$

is a fundamental matrix of the homogeneous equation corresponding to (2.8.4).

According to (2.7.6) (U replaced by V)

$$V^*(x,\bar{\lambda})\, K\, V(x,\lambda) = \begin{pmatrix} M(\lambda) & I_n \\ I_n & 0 \end{pmatrix} U^*(x,\bar{\lambda})\, K\, U(x,\lambda) \begin{pmatrix} M(\lambda) & I_n \\ I_n & 0 \end{pmatrix} = \tag{2.8.6}$$

$$= \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Here $M(\bar{\lambda}) = M(\lambda)^*$ has been observed.

We seek a solution of the initial-value problem (2.8.4 - 5) of the form

$$\hat{g}(x,\lambda) = V(x,\lambda)\, \hat{\eta}(x)\,,$$

where $\hat{\eta}(x)$ is an n-vector function to be found. By differentiation

$$K\hat{g}'(x) = (\lambda |R|(x) + S(x))\hat{g}(x) + KV(x,\lambda)\hat{\eta}'(x)\,.$$

This agrees with (2.8.4) if

$$KV(x,\lambda)\hat{\eta}'(x) = |R|(x)\hat{f}(x)$$

or

$$\hat{\eta}(x) = \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \end{pmatrix} - \int_x^b (KV(t,\lambda))^{-1} |R|(t)\hat{f}(t)dt\,,$$

where $(\hat{c}_1, \hat{c}_2)^T$ is a constant 2n-vector.

$$\hat{g}(x,\lambda) = V(x,\lambda)\left\{ \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \end{pmatrix} - \int_x^b (KV(t,\lambda))^{-1} |R|(t)\hat{f}(t)dt \right\},$$

and $\hat{g}(x,\lambda) \in L^2_R$ if and only if $\hat{c}_2 = 0$. Indeed

$$V(x,\lambda) \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \end{pmatrix} = \begin{pmatrix} \psi(x,\lambda;1) & \varphi(x,\lambda;1) \\ \psi(x,\lambda;2) & \varphi(x,\lambda;2) \end{pmatrix} \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \end{pmatrix} = \begin{pmatrix} \psi(x,\lambda) & \varphi(x,\lambda) \end{pmatrix} \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \end{pmatrix},$$

and we have

$$\int_0^b \hat{c}_1^* \psi(x,\lambda)^* |R|(x)\psi(x,\lambda)\hat{c}_1 dx < \infty$$

whereas $\varphi(x,\lambda)\hat{c}_2$ belongs to L^2_R if and only if $\hat{c}_2 = 0$ (compare p.2.7.3-4)

Therefore

$$\hat{g}(x,\lambda) = V(x,\lambda)\left\{\begin{pmatrix}\hat{c}_1\\0\end{pmatrix} - \int_x^b (KV(t,\lambda))^{-1}|R|(t)\hat{f}(t)dt\right\}. \qquad (2.8.7)$$

Now we compute $\hat{c}_1$ such that the solution (2.8.7) satisfies the initial-value problem (2.8.4 - 5). Put

$$\hat{F} = \begin{pmatrix}\hat{F}_1\\\hat{F}_2\end{pmatrix}, \text{ where } \hat{F} = \int_0^b (KV(t,\lambda))^{-1}|R|(t)\hat{f}(t)dt$$

As

$$\hat{g}(0,\lambda) = V(0,\lambda)\left\{\begin{pmatrix}\hat{c}_1\\0\end{pmatrix} - \begin{pmatrix}\hat{F}_1\\\hat{F}_2\end{pmatrix}\right\},$$

then by (2.8.5) we find

$$V(0,\lambda)\left\{\begin{pmatrix}\hat{c}_1\\0\end{pmatrix} - \begin{pmatrix}\hat{F}_1\\\hat{F}_2\end{pmatrix}\right\} = \begin{pmatrix}(\sin\alpha)\ \hat{c}\\ \tilde{I}_n(\cos\alpha)\ \hat{c}\end{pmatrix}$$

with an arbitrary n-vector $\hat{c}$.

According to

$$V(0,\lambda) = U(0,\lambda)\begin{pmatrix}M(\lambda) & I_n\\ I_n & 0\end{pmatrix} = \begin{pmatrix}(\sin\alpha & (\cos\alpha)\\ \tilde{I}_n(\cos\alpha) & -\tilde{I}_n(\sin\alpha)\end{pmatrix}\begin{pmatrix}M(\lambda) & I_n\\ I_n & 0\end{pmatrix}$$

one has

$$U(0,\lambda)\begin{pmatrix}M(\lambda) & I_n\\ I_n & 0\end{pmatrix}\left\{\begin{pmatrix}\hat{c}_1\\0\end{pmatrix} - \begin{pmatrix}\hat{F}_1\\\hat{F}_2\end{pmatrix}\right\} = \begin{pmatrix}(\sin\alpha)\ \hat{c}\\ (\cos\alpha)\ \hat{c}\end{pmatrix}.$$

This implies

$$\begin{pmatrix}M(\lambda) & I_n\\ I_n & 0\end{pmatrix}\left\{\begin{pmatrix}\hat{c}_1\\0\end{pmatrix} - \begin{pmatrix}\hat{F}_1\\\hat{F}_2\end{pmatrix}\right\} = \begin{pmatrix}(\sin\alpha) & (\cos\alpha)\ \tilde{I}_n\\ (\cos\alpha) & -(\sin\alpha)\ \tilde{I}_n\end{pmatrix}\begin{pmatrix}(\sin\alpha)\ \hat{c}\\ \tilde{I}_n(\cos\alpha)\ \hat{c}\end{pmatrix} = \begin{pmatrix}\hat{c}\\0\end{pmatrix}$$

or

$$\begin{pmatrix}\hat{c}_1\\0\end{pmatrix} - \begin{pmatrix}\hat{F}_1\\\hat{F}_2\end{pmatrix} = \begin{pmatrix}0 & I_n\\ I_n & -M(\lambda)\end{pmatrix}\begin{pmatrix}\hat{c}\\0\end{pmatrix}.$$

That is $\hat{c}_1 = \hat{F}_1$ and $-\hat{F}_2 = \hat{c}$.

Now the unique solution of the initial-value problem (2.8.4 - 5) can be written

$$\hat{g}(x,\lambda) = V(x,\lambda)\left\{\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \int_0^b (K\,V(t,\lambda))^{-1}\,|R|(t)\hat{f}(t)dt - \right.$$

$$\left. - \int_x^b (K\,V(t,\lambda))^{-1}\,|R|(t)\hat{f}(t)dt \right\} . \qquad (2.8.8)$$

From (2.8.6) one obtains

$$(K\,V(x,\lambda))^{-1} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} V(x,\lambda)^* = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \left(U(x,\bar{\lambda}) \begin{pmatrix} M(\bar{\lambda}) & I_n \\ I_n & 0 \end{pmatrix} \right)^* =$$

$$= \begin{pmatrix} I_n & 0 \\ -M(\lambda) & -I_n \end{pmatrix} U(x,\bar{\lambda})^* .$$

Again $M(\lambda)^* = M(\bar{\lambda})$ has been used. As

$$V(x,\lambda)\begin{pmatrix} I_n & 0 \\ -M(\lambda) & -I_n \end{pmatrix} = U(x,\lambda)\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

the solution (2.8.8) can be written

$$\hat{g}(x,\lambda) = V(x,\lambda)\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \int_0^b U(x,\lambda)^*|R|(t)\hat{f}(t)dt -$$

$$- U(x,\lambda)\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \int_x^b U(x,\bar{\lambda})^*|R|(x)\hat{f}(t)dt . \qquad (2.8.9)$$

With $|R_1|(t) = \begin{pmatrix} |r(t)| & 0 \cdots 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \cdots 0 \end{pmatrix}$ an $n \times n$-matrix and

$\hat{f}(t) = (f(t), 0, \ldots, 0)^T = (\hat{f}_1(t), \hat{f}_2(t))^T$, $\hat{f}_1(t) = (f(t), 0, \ldots, 0)^T$

which is an n-vector-valued function, and $\hat{g}(x,\lambda) = (\hat{g}_1(x,\lambda), \hat{g}_2(x,\lambda))^T$.

From (2.8.9) one obtains

$$\hat{g}_1(x,\lambda) = \psi(x,\lambda;1) \int_0^b \varphi(t,\bar{\lambda};1)^*|R_1|(t)\,\hat{f}_1(t)dt +$$

$$+ \varphi(x,\lambda;1) \int_x^b \theta(t,\bar{\lambda};1)^*|R_1|(t)\,\hat{f}_1(t)dt - \qquad (2.8.10)$$

$$- \theta(x,\lambda;1) \int_x^b \varphi(t,\bar{\lambda};1)^*|R_1|(t)\,\hat{f}_1(t)dt .$$

Here use has been made of

$$\psi(x,\lambda) = \begin{pmatrix} \psi(x,\lambda;1) \\ \psi(x,\lambda;2) \end{pmatrix}, \quad \varphi(x,\lambda) = \begin{pmatrix} \varphi(x,\lambda;1) \\ \varphi(x,\lambda;2) \end{pmatrix} .$$

From (2.7.9), the unique solution (2.8.10) is finally obtained

$$\hat{g}_1(x,\lambda) = \psi(x,\lambda;1) \int_0^x \varphi(t,\bar{\lambda};1)^* |R_1|(t)\ \hat{f}_1(t)dt +$$

$$+ \varphi(x,\lambda;1) \int_x^b \psi(t,\bar{\lambda};1)^* |R_1|(t)\ \hat{f}_1(t)dt\ .$$

The first element of $\hat{g}_1(x,\lambda)$ is

$$g(x,\lambda) = \sum_{k=0}^{n-1} (\psi_k(x,\lambda) \int_0^x \varphi_k(t,\lambda) f(t) |r|(t)dt + \varphi_k(x,\lambda) \int_x^b \psi_k(t,\lambda) f(t) |r|(t)dt)\ ,$$

i.e. the asserted relation (2.8.3) is obtained.

2.9. The second order case

This section is divided into two parts 2.9 (a) and 2.9 (b). In 2.9 (a) an explicit condition on the coefficients of a second order equation is given which implies the basic assumption IV, hence makes the definition of associated operators possible. In 2.9 (b) spectral properties of these operators are considered. In view of the basic importance of assumption IV, an elementary original treatment is presented in 2.9 (a) which differs from the shorter discussion in K. Daho, H. Langer [6].

2.9 (a). Conditions that with a second order equation an operator can be associated.

The left hand side of the equation

$$-(\tfrac{1}{p}y')' + q\,y = |r|f \quad \text{a.e. on } [0,b) \tag{2.9.1}$$

is a differential expression $M(y)$ of the standard form in Section 2.1 with $n=1$, $p_0=\frac{1}{p}$ and $p_1=q$. The basic assumptions I - III are verbally transferred into:

I. p, q and r are real and belong to $L_{loc}[0,b)$.

II. $\frac{1}{p(x)} > 0$, $q(x) \geq 0$, in both cases a.e. on $[0,b)$.

III. $\int_0^x |r(t)|dt > 0$, $0 < \int_x^b |r(t)|dt \leq \infty$ for $0 < x < b$.

The equation (2.9.1) is again to be understood as the first order system

$$\begin{aligned} (y^{[1]})' &= qy - |r|f, \quad \text{a.e. on } [0,b) \\ y' &= py^{[1]}, \quad \text{a.e. on } [0,b) \end{aligned} \tag{2.9.2}$$

with absolutely continuous functions y and $y^{[1]}$.

We consider the slightly more general situation obtained by relaxing the condition II by allowing the function $p(x)$ to vanish on a set positive measure:

II'. $p(x) \geq 0$, $q(x) \geq 0$ a.e. on $[0,b)$

As assumption IV′ (instead of IV) we introduce

IV′. For any subinterval Δ of $[0,b)$

$$\int_\Delta p(x)dx = 0 \quad \text{implies} \quad \int_\Delta |r(x)|dx = 0 .$$

Observe that this assumption is in particular fulfilled if $p(x) > 0$ a.e.

Under the assumptions I, II′, III, the assumption IV of Section 2.2 is a consequence of IV′. This follows from

LEMMA 2.9.1. Under the assumptions I, II′, III, IV′ suppose that for some b': $0 \le b' < b$ we have

$$y(b') = 0, \quad \int_{b'}^{b} |y(x)|^2 |r(x)|dx = 0, \quad -(\tfrac{1}{p} y')' + q y = |r| f$$

with some function f: $\int_0^{b''} |f(x)|^2 |r(x)|dx < \infty$ for each b'': $0 < b'' < b$.

Then

$$\int_{b'}^{b} |f(x)|^2 |r(x)|dx = 0, \quad y(x) = 0 \text{ and } y^{[1]}(x) = 0 \quad (x \in [b',b)) .$$

If we have $\int_0^b |y(x)|^2 |r(x)|dx = 0$ with some continuous function y on $[0,b)$, the first part of III implies $y(0) = 0$. Choosing a sequence (b_m), such that $b_m \uparrow b$ and $\int_x^{b_m} |r(t)|dt > 0$ for all $x < b_m$, $m = 1,2,\ldots$, it follows, that the assumptions I, II′, III, IV′ imply IV.

Proof of the lemma. By separation of real and imaginary parts, it suffices to prove the lemma for real functions y and f.

The first aim is to prove that $y(x) = 0$ for $b' \le x < b$. For a proof by contradiction, assume that there exists a point t in $[b',b)$ such that $y(t) \neq 0$. Because of the continuity of y there exists a subinterval $\Delta = (x_1,x_2)$ of $[b',b)$ such that $y(x) \neq 0$ on Δ. The assumption $\int_b^b |r(x)|\,|y(x)|^2 dx = 0$ implies that $\int_\Delta |r(x)|\,|y(x)|^2 dx = 0$, and $r(x) = 0$ a.e. on Δ. Since $y(b') = 0$ the point b' is not included in Δ, and because of III the interval cannot be extended to b. It is therefore a proper subinterval of $[b',b)$, and is finite. By the continuity of y the interval Δ can be enlarged if $y(x_1) \neq 0$ or $y(x_2) \neq 0$. Therefore, it

can be assumed that $y(x_1) = y(x_2) = 0$ while $y(x) \neq 0$ in $x_1 < x < x_2$. Since $r(x) = 0$ a.e. on Δ, the first equation in (2.9.2) reduces to $y^{[1]\prime} = qy$ a.e. on Δ. From $q(x) \geq 0$ a.e. and from $y(x) \neq 0$ on Δ it follows that $y^{[1]\prime}$ has the same sign as $y(x)$ a.e. on Δ. Assume this sign positive (the case $y(x) < 0$ is similarly treated). Then $y^{[1]}(x)$ is non-decreasing on Δ. Because of the continuity of y the value $y(x)$ assumes a maximum $y(x_0) = c$ on Δ at an innerpoint x_0 of this interval. The continuity also implies that x_0 can be taken as the minimum of all points with these properties. Hence

$$y(x_0) - y(x) > 0 \quad \text{if} \quad x < x_0, \quad x \in \Delta .$$

According to the second equation (2.9.2)

$$\int_x^{x_0} p(t) y^{[1]}(t)dt = y(x_0) - y(x) > 0 \quad \text{for} \quad x < x_0, \quad x \in \Delta .$$

As $p(x) \geq 0$ a.e. there exists a point v between x and x_0 such that $p(v)$ and $y^{[1]}(v)$ are positive. But $y^{[1]}(x)$ is non-decreasing on Δ so that

$$y^{[1]}(x) \geq y^{[1]}(v)$$

for $x > v$, $x \in \Delta$, and is à fortiori positive for $x > x_0$, $x \in \Delta$. For such a point x

$$y(x) - y(x_0) = \int_{x_0}^{x} p(t)\, y^{[1]}(t)dt \geq 0$$

since the integrand is non-negative. This, together with the property $y(x) \leq y(x_0)$, shows that $y(x)$ must be a constant for $x > x_0$. But this constant must be 0 since $y(x_2) = 0$, which is a contradiction to $y(x) \neq 0$ $(x \in \Delta)$.

Having shown the vanishing of y on $[b',b)$ the equations (2.9.2) take the form

$$\begin{aligned} y^{[1]\prime} &= -|r|f \quad \text{a.e. on } [0,b) \\ p\, y^{[1]} &= 0 \quad \text{a.e. on } [0,b) \end{aligned} \tag{2.9.2'}$$

To prove also that $y^{[1]}(x) = 0$ on $[b',b)$, we write this interval as the union of the disjoint sets E_0 and E_+:

$x \in E_0$ if there is an open subinterval Δ of $[b',b)$ containing x such that $\int_\Delta p(t)dt = 0$,

$x \in E_+$ if for any open subinterval Δ containing x, $\int_\Delta p(t)dt > 0$.

If $x \in E_+$, then $\int_\Delta p(t)dt > 0$ for any subinterval Δ containing x. The second equation in (2.9.2′) implies $y^{[1]}(x) = 0$ on Δ. Therefore $y^{[1]}$ must vanish on E_+.

If $x \in E_0$, we choose a subinterval Δ such that $x \in \Delta$ and $p=0$ a.e. on Δ. This implies $r=0$ a.e. on Δ according to IV′. Hence, the first equation in (2.9.2′) reduces to $y^{[1]\prime} = 0$ a.e. on Δ, and $y^{[1]}(x)$ is constant on this interval. However, the subinterval Δ can be taken to be maximal so that its endpoints x_1 and x_2 belong to E_+. Observe that $x_2 < b$: Indeed, $x_2 = b$ implies $p=0$ a.e. on (x_1,b), hence by IV′, $r=0$ a.e. on (x_1,b), a contradiction with III. As $y^{[1]}$ is constant on Δ, continuous x_1 and x_2 and $y^{[1]}(x_1) = y^{[1]}(x_2) = 0$, it follows that $y^{[1]}$ vanishes on Δ. As a consequence, $y^{[1]} = 0$ on $[b',)$.

COROLLARY 2.9.2. *A function* y *on* $[0,b)$ *which satisfies the differential equation*

$$-(\tfrac{1}{p}y')' + qy = \lambda|r|y \quad \text{a.e. on } [0,b), \tag{2.9.3}$$

with $\lambda \in \mathbb{C}$ *and the condition* $\int_0^b |r(x)||y(x)|^2dx = 0$ *is identical zero.*

2.9 (b). *Spectral properties of the operators* B_α.

For arbitrary complex λ let $\varphi_\alpha(\cdot,\lambda)$ and $\theta_\alpha(\cdot,\lambda)$ $(\alpha \in [0,\pi))$ denote the solutions of the equation (2.9.3) with initial conditions (2.7.4) at zero, i.e.

$$\varphi_\alpha(0,\lambda) = \sin\alpha \qquad \theta_\alpha(0,\lambda) = \cos\alpha$$
$$\varphi_\alpha^{[1]}(0,\lambda) = \cos\alpha \qquad \theta_\alpha^{[1]}(0,\lambda) = -\sin\alpha . \tag{2.9.4}$$

We suppose now that the endpoint b is <u>regular</u> for the equation (2.9.1).

Every selfadjoint extension B_α is determined by a boundary condition of the form (see 2.6.6)

$$y(b)\cos\beta + y^{[1]}(b)\sin\beta = 0, \quad \beta \in [0,\pi) . \tag{2.9.5}$$

Consider the corresponding coefficient of H. Weyl, that is the function $m_{\alpha,\beta}(\cdot,b)$ of the complex parameter λ, such that $\psi_{\alpha,\beta}(\cdot,\lambda) = \theta_\alpha(\cdot,\lambda) + m_{\alpha,\beta}(\lambda)\,\varphi_\alpha(\cdot,\lambda)$ is a solution of the initial-value problem (2.9.3), (2.9.5). Then

$$m_{\alpha,\beta}(\lambda,b) = -\frac{\theta_\alpha(b,\lambda)\cos\beta + \theta_\alpha^{[1]}(b,\lambda)\sin\beta}{\varphi_\alpha(b,\lambda)\cos\beta + \varphi_\alpha^{[1]}(b,\lambda)\sin\beta}, \quad \alpha,\beta \in [0,\pi) . \tag{2.9.6}$$

Similary to (2.7.7-8) we have

$$m_{\alpha,\beta}(\lambda,b) = \frac{-\sin(\alpha_1-\alpha) + m_{\alpha_1,\beta}(\lambda,b)\cos(\alpha_1-\alpha)}{\cos(\alpha_1-\alpha) + m_{\alpha_1,\beta}(\lambda,b)\sin(\alpha_1-\alpha)} \tag{2.9.7}$$

with α_1, α, $\beta \in [0,\pi)$.

Some well-known properties of the function

$$m_{\frac{\pi}{2},0}(\cdot,b) = -\frac{\theta_{\pi/2}(b,\cdot)}{\varphi_{\pi/2}(b,\cdot)}$$

are added.

LEMMA 2.9.3. The function $m_{\frac{\pi}{2},0}(\cdot) = m_{\frac{\pi}{2},0}(\cdot,b)$ is real for real λ, and has the following properties

$$m_{\frac{\pi}{2},0}(\cdot) \in N_0(\mathbb{C}), \quad m_{\frac{\pi}{2},0}(\lambda) \text{ if } \lambda < 0,$$

$$\lim_{\lambda \to -\infty} m_{\frac{\pi}{2},0}(\lambda) = 0.$$

Proof. The first statement is contained in Lemma 2.7.1. To prove the second and third relation we show, that for $\lambda < 0$ the following relations hold true:

$$\varphi_{\pi/2}(x,\lambda) \geq 1, \quad \varphi^{[1]}_{\pi/2}(x,\lambda) \geq 0, \quad 0 \leq x < b \tag{2.9.8}$$

$$\theta_{\pi/2}(x,\lambda) \leq 0, \quad \theta^{[1]}_{\pi/2}(x,\lambda) \leq 0, \quad 0 \leq x < b \tag{2.9.9}$$

$$\lim_{\lambda \to -\infty} \varphi_{\pi/2}(x,\lambda) = \infty, \quad \lim_{\lambda \to -\infty} \theta_{\pi/2}(x,\lambda) = -\infty, \quad 0 < x < b. \tag{2.9.10}$$

For the proof of (2.9.8) suppose that $\varphi_{\pi/2}(\cdot,\lambda)$ has a (smalles) zero $x_0 \in (0,b)$. Then, according to (2.9.3), one obtains

$$\varphi^{[1]}_{\pi/2}(x,\lambda) \geq 0 \quad \text{a.e. on} \quad 0 \leq x \leq x_0$$

and

$$\varphi'_{\pi/2}(x,\lambda) \geq 0 \quad \text{a.e. on} \quad 0 \leq x \leq x_0$$

which contradicts $\varphi_{\pi/2}(0,\lambda) = 1$, $\varphi_{\pi/2}(x_0,\lambda) = 0$, and (2.9.8) follows.

Moreover, from (2.9.3) and (2.9.8)

$$\varphi^{[1]}_{\pi/2}(x,\lambda) \geq |\lambda| \int_0^x |r(s)|\,ds$$

and the first relation in (2.9.10) follows.

The proof of (2.9.9) and the second relation in (2.9.10) are analogous. The second relation in the lemma follows from (2.9.8) and (2.9.9). To prove the third, consider

$$\frac{d}{dx}\left(-\frac{\theta_{\pi/2}(x,\lambda)}{\varphi_{\pi/2}(x,\lambda)}\right) = \frac{\theta_{\pi/2}(x,\lambda)\,\varphi'_{\pi/2}(x,\lambda) - \theta'_{\pi/2}(x,\lambda)\,\varphi_{\pi/2}(x,\lambda)}{\varphi^2_{\pi/2}(x,\lambda)} =$$

$$= p(x)\,/\varphi^2_{\pi/2}(x,\lambda)$$

(according to $\det U(x,\lambda) = -1$, c.f. Section 2.7). Hence

$$m_{\frac{\pi}{2},0}(\lambda,b) = \int_0^b \frac{p(x)}{\varphi^2_{\pi/2}(x,\lambda)}\,dx$$

and the conclusion follows from (2.9.8) and (2.9.10).

In the terminology of I.S. Kac and M.G. Krein [14, §5], the function $m_{\frac{\pi}{2},0}(\cdot,b)$ belongs to the class (S). As a consequence, it admits an integral representation

$$m_{\frac{\pi}{2},0}(\lambda,b) = \int_{0^-}^{\infty} \frac{d\sigma(t)}{t-\lambda} \qquad (\lambda \in (-\infty,0)$$

with a non-decreasing function σ on $[0,\infty)$. It follows that

$$\frac{d}{d\lambda}\left(m_{\frac{\pi}{2},0}(\lambda,b)\right) > 0 \quad \text{when} \quad \lambda \in (-\infty,0).$$

Suppose now that b is <u>singular</u> and the equation (2.9.6) is of the <u>limit type</u> (1,1) <u>at</u> b (limit point at b). The coefficient of Weyl $m_{\alpha,\beta}(\cdot,b')$, $b' < b$ converges for $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ to a holomorphic function m_α in $\mathbb{C}_+ \cup \mathbb{C}_-$, if $b' \uparrow b$ which is independent of β. In this case the relation (2.9.7) holds true for m_α and m_{α_1} (instead of $m_{\alpha,\beta}(\cdot,b')$ and $m_{\alpha_1,\beta}(\cdot,b')$. As a consequence one obtains from (2.9.7) that

$$m_\alpha(\lambda) = \frac{-\sin(\beta-\alpha) + m_\beta(\lambda)\cos(\beta-\alpha)}{\cos(\beta-\alpha) + m_\beta(\lambda)\sin(\beta-\alpha)}. \qquad (2.9.11)$$

This relation had been proved earlier in another way (see (2.7.10)).

The function $m_{\pi/2}(\lambda)$ still has the properties of $m_{\frac{\pi}{2},0}(\lambda)$ in Lemma 2.9.3, i.e.

$$m_{\pi/2}(\cdot) \in N_0(\mathbb{C}), \quad m_{\pi/2}(\lambda) > 0 \quad \text{if} \quad \lambda < 0, \qquad (2.9.12)$$

$$\lim_{\lambda \to -\infty} m_{\pi/2}(\lambda) = 0.$$

Indeed, choose a sequence (b_n): $b_n \uparrow b$, then $m_{\frac{\pi}{2},0}(\cdot,b_n)$ admits a representation

$$m_{\frac{\pi}{2},0}(\lambda,b_n) = \int_{0^-}^{\infty} \frac{d\sigma_n(t)}{t-\lambda} \qquad (\lambda \in (-\infty,0))$$

with a non-decreasing function σ_n on $[0,\infty)$.

The convergence of $m_{\frac{\pi}{2},0}(i,b_n)$ implies

$$\int_{0^-}^{\infty} \frac{d\sigma_n(t)}{1+t} \le C < \infty, \quad n = 1,2,3,\dots,$$

where C is some positive constant. Therefore

$$m_{\pi/2}(\lambda) = \int_{0^-}^{\infty} \frac{d\sigma(t)}{t-\lambda}$$

with a non-decreasing function σ on $[0,\infty)$ and

$$\int_{0^-}^{\infty} \frac{d\sigma(t)}{1+t} < C < \infty.$$

In the following lemma we establish some properties of the operator B_α in the general case, that is, if B_α is not necessarily supposed to be selfadjoint.

LEMMA 2.9.4. <u>There exists an</u> $\alpha_0 \in (\frac{\pi}{2},\pi)$ <u>with the following properties</u>:

(1) $B_\alpha \ge 0$ <u>if</u> $\alpha \in [0,\alpha_0]$.

(2) <u>When</u> $\alpha \in (\alpha_0,\pi)$, <u>the bilinear form</u> $(B_\alpha u,v)$ $(u,v \in D_\alpha)$ <u>has exactly one negative square</u> (see Section 1.1).

<u>Proof</u>. Consider the set

$$D'_\alpha = \{y \in \tilde{D}:\ y \equiv 0 \text{ in a neighbourhood of } b \text{ and } y(0)\cos\alpha - y^{[1]}(0)\sin\alpha = 0\}.$$

If $u \in D'_\alpha$ one obtains

$$(B_\alpha u,u) = \int_0^b (p|u^{[1]}|^2 + q|u|^2)dx + u^{[1]}(0)\,\overline{u(0)} =$$

$$= \begin{cases} \int_0^b (p|u^{[1]}|^2 + q|u|^2)dx & \text{if } \alpha = 0, \qquad (*) \\ \int_0^b (p|u^{[1]}|^2 + q|u|^2)dx + \cot\alpha\,|u(0)|^2 & \text{if } \alpha \in (0,\pi). \end{cases}$$

Therefore $B_\alpha \geq 0$ if $\alpha \in [0, \frac{\pi}{2})$ and the bilinear form $(B_\alpha u,v)$, $u,v \in D'_\alpha$, $\alpha \in (\frac{\pi}{2}, \pi)$ has at most one negative square.

Next it will be shown that $B_{\alpha_1} \geq 0$ implies $B_\alpha \geq 0$ for $\alpha \in (0,\alpha_1)$. Suppose the contrary, i.e. that there exists an $\alpha_2 < \alpha_1$ and a $v \in D'_{\alpha_2}$ such that $(B'_{\alpha_2} v,v) < 0$.

Choose $b'' < b$ such that $v(x)$ vanishes if $x > b''$ and Condition III is satisfied for b'' instead of b. Then the operators $B''_{\alpha_1}, B''_{\alpha_2}$ corresponding to the restriction of the differential expression to $[0,b'']$, satisfy the same condition as $B_{\alpha_1}, B_{\alpha_2}$ i.e. $B''_{\alpha_1} \geq 0$, $(B''_{\alpha_2} v,v) < 0$ and the restricted differential operators are regular. Therefore, without loss of generality, one may suppose that b is regular. Let $B_{\alpha,0}$ denote the selfadjoint extension of B_α belonging to the boundary condition $u(b) = 0$. From (*) it follows that

$$B_\alpha \geq 0 \text{ is equivalent to } B_{\alpha,0} \geq 0 \tag{2.9.13}$$

The operator $B_{\alpha_1,0}$ $(B_{\alpha_2,0})$ is non-negative (has at least one negative eigenvalue). Therefore, from (2.9.7) the equation

$$m_{\frac{\pi}{2},0}(\lambda,b) = -\cot(\tfrac{\pi}{2} - \alpha_1) \quad (m_{\frac{\pi}{2},0}(\lambda,b) = -\cot(\tfrac{\pi}{2} - \alpha_2))$$

does not have one (has at least one) negative solution λ. But as the function $m_{\frac{\pi}{2},0}(\lambda,b)$ is positive and non-decreasing (see Lemma 2.9.3, this gives a contradiction if $\alpha_1 > \alpha_2$. And hence $B_{\alpha_1} \geq 0$ implies $B_\alpha \geq 0$ with $\alpha \in (0,\alpha_1)$.

The proof of the lemma will be complete if one shows that

$$\alpha_0 = \sup \{\alpha: B_\alpha \geq 0\} \tag{2.9.14}$$

has the properties $\alpha_0 < \pi$ and $B_{\alpha_0} \geq 0$.

To see the first inequality, suppose $B_\alpha \geq 0$ for all $\alpha \in [0,\pi)$. Then the same is true for the operator B''_α with a certain $b'' < b$, i.e. one can again suppose that b is regular.

Now (2.9.13) implies $B_{\alpha,0} \geq 0$ for all $\alpha \in [0,\pi)$ which is impossible because of (2.9.7).

Assume there is a $u_0 \in D'_{\alpha_0}$ such that $(B_{\alpha_0}u_0,u_0) < 0$. One can again suppose that b is regular. Then $B_{\alpha_0,0}$ has at least one negative eigenvalue, i.e. the equation

$$m_{\frac{\pi}{2},0}(\lambda,b) = -\cot(\tfrac{\pi}{2} - \alpha_0)$$

has at least one negative solution. Then the same is true for $B_{\alpha,0}$ if only 0 is sufficiently close to α_0. This is a contradiction to (2.9.13).

The following results concern only the case when b is singular and the differential equation (2.9.1) is in the limit (1,1) case at b. This is equivalent to the selfadjointness of the operators B_α, $\alpha \in [0,\pi)$. Under this condition the proof of Lemma 2.9.4 can be much simplified (see [6], Lemma 2.4). Then α_0 is the solution in $[\frac{\pi}{2},\pi)$ of the equation

$$\lim_{\lambda \uparrow 0} m_{\pi/2}(\lambda) = -\tan\alpha_0 \quad (\alpha_0 = \tfrac{\pi}{2} \text{ if this limit is } \infty). \qquad (2.9.15)$$

Indeed, the operator B_α has a negative eigenvalue λ_α if and only if the function m_α has a negative pole λ_α. From (2.9.11) with $\beta = \frac{\pi}{2}$ and the monotonicity of the function $m_{\pi/2}$ on the negative axis, it follows that this is the case if and only if $\alpha \in [\frac{\pi}{2},\pi)$ is greater than the solution $\alpha_0 \in [\frac{\pi}{2},\pi)$ of the equation (2.9.15).

Moreover, in this case λ_α is the negative solution of the equation

$$m_{\pi/2}(\lambda) + \tan\alpha = 0 \qquad \alpha \in (\alpha_0,\pi).$$

It follows that λ_α $(\alpha \in (\alpha_0,\pi))$ is a non-decreasing and analytic function of α with the properties

$$\lim_{\alpha \downarrow \alpha_0} \lambda_\alpha = 0 \quad \text{and} \quad \lim_{\alpha \uparrow \pi} \lambda_\alpha = -\infty.$$

It should be mentioned, that similar arguments have been used by M.G. Krein [14, §5].

LEMMA 2.9.5. Suppose b is singular and the equation $-(\frac{1}{p}y')' + qy = |r|f$ is in the limit point-case at b. If 0 is an eigenvalue of the self-

adjoint operator B_α in the Hilbert space $L_r^2[0,b)$, $\alpha \in [0,\pi)$, then $\alpha = \alpha_0$.

Proof. Suppose there exists an $\alpha > \alpha_0$ such that $0 \in \sigma_p(B_\alpha)$, i.e. $B_\alpha u_\alpha = 0$, $u_\alpha \neq 0$. Since the operator B_α has at most one negative eigenvalue λ_α (if $\alpha \in (\alpha_0,\pi)$), the associated coefficient of Weyl $m_\alpha(\lambda)$ admits a representation

$$m_\alpha(\lambda) = \frac{c_0}{\lambda_\alpha - \lambda} - \frac{c_1}{\lambda} + \tilde{m}(\lambda)$$

where

$$c_0 = \begin{cases} 0 \ , & \text{if } \alpha \in [0,\alpha_0] \\ > 0 \ , & \text{if } \alpha \in (\alpha_0,\pi) \end{cases}$$

$c_1 > 0$ and $\tilde{m}(\lambda)$ has the property $\lim\limits_{\lambda \uparrow 0} \lambda \tilde{m}(\lambda) = 0$. According to (2.9.11), consider

$$m_\beta(\lambda) = \frac{\sin(\beta-\alpha) + m_\alpha(\lambda)\cos(\beta-\alpha)}{\cos(\beta-\alpha) - m_\alpha(\lambda)\sin(\beta-\alpha)} \ .$$

The poles of $m_\beta(\lambda)$ on the real axis are solution of

$$\cot(\beta-\alpha) = m_\alpha(\lambda) \ .$$

Multiplying the above equation by λ on both sides and using the representation of $m_\alpha(\lambda)$, one obtains

$$\lambda \cot(\beta-\alpha) = \lambda m_\alpha(\lambda) = \lambda\left(\frac{c_0}{\lambda_\alpha - \lambda} - \frac{c_1}{\lambda} + \tilde{m}(\lambda)\right) .$$

Letting $\lambda \to 0$ one gets

$$\lambda \cot(\beta-\alpha) = -(o(1) + c_1)$$

or

$$-\frac{\lambda}{o(1)+c_1} = \tan(\beta-\alpha) \ .$$

If β is sufficiently close to α, $\beta \geq \alpha$, this equation has a negative solution λ, which is arbitrarily small. Therefore, $\alpha > \alpha_0$ gives a contradiction, since for these β, $\lambda_\beta < \lambda_\alpha$ and the operator B_β cannot have two negative eigenvalues.

The inequality $\alpha < \alpha_0$ is also impossible since

$$\alpha_0 = \sup\{\alpha: \ \sigma(B_\alpha) \geq 0, \ \tfrac{\pi}{2} \leq \alpha < \pi\} \ .$$

Thus $\alpha = \alpha_0$.

3. QUASI-DIFFERENTIAL OPERATORS WITH INDEFINITE WEIGHT FUNCTION

3.1. Operators associated with $M(y) = rf$. The r-limit (n,n) property.

In the sequel, the coefficients in $M(y) = rf$ are always assumed to satisfy the basic assumptions I - IV and, moreover, if the contrary is not explicitly stated, it is supposed that b be singular and the equation $M(y) = |r|f$ is in limit (n,n) case at b. As in Section 2.6, α is a real symmetric[1] $(n\times n)$-matrix, $0 \le \alpha < \pi I_n$. In the Krein space L^2_r the positive definite scalar product, the indefinite scalar product and the operator J are given by (1.1.4), (1.1.5) and (1.1.6) respectively. It is always assumed that the scalar product (1.1.5) is indefinite, i.e. r changes sign or $J \neq I$.

In the Krein space L^2_r, the operators

$$\tilde{A} = J\tilde{B}, \quad A = JB, \quad A_\alpha = JB_\alpha, \quad A'_\alpha = JB'_\alpha$$

are considered. Their domains coincide with the domains $\tilde{D}$, D, D_α and D'_α of $\tilde{B}$, B, B_α and B'_α respectively (see Sections 2.5, 2.6).

Evidently, under the above conditions A is J-symmetric, A_α is J-selfadjoint and the following relations hold true:

$$[\tilde{A}f, f] = (\tilde{B}f, f) \qquad (f \in \tilde{D}),$$

$$[Af, f] = (Bf, f) \qquad (f \in D), \tag{3.1.1}$$

$$[A_\alpha f, f] = (B_\alpha f, f) \qquad (f \in D_\alpha). \tag{3.1.2}$$

The resolvent set of a selfadjoint operator in a Hilbert space is non-empty. This is in general not true for a J-selfadjoint operator, even if it is J-nonnegative (see Bognár [4], H. Langer [26]). However, if b is regular and $A_{\alpha,\beta} = JB_{\alpha,\beta}$ (see Section 2.6) is J-selfadjoint, then $\rho(A_{\alpha,\beta}) \neq \emptyset$. This will be proved only for the second-order case; the proof for the general case is similar (cf. H. Langer [28], Lemma 2.4).

For arbitrary complex λ, let $\chi_\alpha(\cdot,\lambda)$ and $\varphi_\alpha(\cdot,\lambda)$ be the solutions of

$$-(\tfrac{1}{p}y')' + qy = \lambda r y \quad \text{on} \quad [0,b) \tag{3.1.3}$$

---

1) In the sequel, by a symmetric $(n\times n)$-matrix we always mean a real and symmetrix matrix.

with initial conditions

$$\varphi_\alpha(0,\lambda) = \sin\alpha, \qquad \chi_\alpha(0,\lambda) = \cos\alpha,$$
$$\varphi_\alpha^{[1]}(0,\lambda) = \cos\alpha, \qquad \chi_\alpha^{[1]}(0,\lambda) = -\sin\alpha. \tag{3.1.4}$$

Let $A_{\alpha,\beta}$ be the regular operator $A_{\alpha,\beta} = JB_{\alpha,\beta}$, $\alpha,\beta \in [0,\pi)$. More precisely, $A_{\alpha,\beta}$ is the restriction of $\tilde{A}$ to the set

$$D(A_{\alpha,\beta}) = \{u \in \tilde{D}: \cos\alpha u(0) - \sin\alpha u^{[1]}(0) = 0, \quad \cos\beta u(b) + \sin\beta u^{[1]}(b) = 0\}. \tag{3.1.5}$$

The corresponding coefficient of Weyl, $w_{\alpha,\beta}$, is a function of λ such that

$$\psi_\alpha(x,\lambda) = \chi_\alpha(x,\lambda) + w_{\alpha,\beta}(\lambda)\,\varphi_\alpha(x,\lambda)$$

is a solution of the initial problem (3.1.3) and $\cos\beta u(b) + \sin\beta u^{[1]}(b) = 0$. If λ is not a pole of $w_{\alpha,\beta}$, then $\lambda \in \rho(A_{\alpha,\beta})$. Indeed, let

$$g(x,\lambda) = \psi_\alpha(x,\lambda)\int_0^x r(t)f(t)\varphi_\alpha(t,\lambda)dt + \varphi_\alpha(x,\lambda)\int_x^b r(t)f(t)\psi_\alpha(t,\lambda)dt$$

with $f \in L^2_r$. Then

$$g^{[1]}(x,\lambda) = \psi_\alpha^{[1]}(x,\lambda)\int_0^x r(t)f(t)\varphi_\alpha(t,\lambda)dt + \varphi_\alpha^{[1]}(x,\lambda)\int_x^b r(t)f(t)\psi_\alpha(t,\lambda)dt$$

and by differentiation

$$g^{[1]\prime}(x,\lambda) = \psi_\alpha^{[1]\prime}(x,\lambda)\int_0^x r(t)f(t)\varphi_\alpha(t,\lambda)dt + \varphi_\alpha^{[1]\prime}(x,\lambda)\int_x^b f(t)r(t)\psi_\alpha(t,\lambda)dt - f(x)r(x).$$

Hence

$$-g^{[1]\prime} + q\,g = \lambda\, r\, g + r\, f.$$

The function g also satisfies the boundary conditions at $x = 0$ and $x = b$ in (3.1.5). But this means $g \in D(A_{\alpha,\beta})$ and $(A_{\alpha,\beta} - \lambda I)\,g = f$.

In the general case when the equation $M(y) = |r|\,f$ on $[0,b)$ is in

limit (n,n) case at b, we have the following

LEMMA 3.1.1. The following statements are equivalent:

(i) There exists a symmetric $(n\times n)$-matrix α, $0 \le \alpha < \pi I_n$, such that $\rho(A_\alpha) \neq \emptyset$.

(ii) There exists a symmetric $(n\times n)$-matrix α, $0 \le \alpha < \pi I_n$ and a complex number λ_0 such that the range $(A_\alpha - \lambda_0 I)D_\alpha$ is closed.

(iii) There exists a complex number λ_0 such that $(A - \lambda_0 I)D$ is closed.

(iv) $\rho(A_\alpha) \neq \emptyset$ for all symmetric $(n\times n)$-matrices α, $0 \le \alpha < \pi I_n$.

Proof. The implications (iv) ==> (i) ==> (ii) are obvious. The implication (ii) ==> (iii) will now be shown. Consider the set $(A - \lambda_0 I)D$ under the assumption (ii). As the factor space $(A_\alpha - \lambda_0 I)D_\alpha / (A - \lambda_0 I)D$ is n-dimensional and $(A_\alpha - \lambda_0 I)D_\alpha$ is closed, the range $(A - \lambda_0 I)D$ must also be closed. It remains to show (iii) ==> (iv). The equation $(A - \lambda_0 I)y = 0$ is equivalent to the initial-value problem:

$$y^{[2n]} = \lambda_0 r y\,, \quad y^{[k]}(0) = 0\,, \quad k = 0,1,\ldots, 2n-1\,, \quad y \in L^2_r\,,$$

which has only the trivial solution $y = 0$. Therefore (iii) implies that the set $r(A)$ of points of regular type[1] of A is non-empty. As in the reasoning about the set D_0' in Section 2.5 one obtains

$$(B_0' y, y) = \sum_{\nu=0}^{n} \int_0^b p_\nu(x)\, |y^{[n-\nu]}(x)|^2 dx \ge 0\,, \quad (y \in D_0') \qquad (3.1.6)$$

and (3.1.1) implies $[Ay,y] \ge 0$, $(y \in D)$, i.e. A is J-nonnegative. Lemma 1.2 of H. Langer [26] implies that for arbitrary λ belonging to the set $r(A)$, the defect of the subspace $(A - \lambda I)D$ is equal to the deficiency index of the non-negative operator B, which is equal to n. Therefore,

---

1) Let T be a closed operator. A complex number λ is a point of regular type of T if and only if the operator $T - \lambda I$ is injective and the range $R(T - \lambda I)$ is closed.

for an arbitrary symmetric $(n\times n)$-matrix α, $0 \le \alpha < \pi I_n$, the range $(A_\alpha - \lambda I)D_\alpha$, $\lambda \in r(A)$, is either L_r^2 or λ is an eigenvalue of A_α. As the second case can take place only for an at most finite number of points λ (by Proposition 1.3 and the separability of L_r^2) and the non-empty set $r(A)$ is open, the conclusion $\rho(A_\alpha) \neq \emptyset$ follows.

COROLLARY 3.1.2. The equivalent statements (i - iv) of Lemma 3.1.1 hold if the range of B is closed.

Proof. The range of B is closed, if and only if the range of A is closed. Therefore, in this case (iii) is satisfied with $\lambda_0 = 0$.

COROLLARY 3.1.3. The equivalent statements (i - iv) of Lemma 3.1.1 hold if one of the following two conditions is satisfied:

(a) $p_n(x) \ge C|r(x)|$ a.e. on $[0,b)$ with a positive constant C.

(b) There exists a value b' in $[0,b)$, such that $r(x)$ does not change its sign if $x > b'$ and $\int_{x_1}^{b'} |r(x)|dx > 0$, $\int_{b'}^{x_2} |r(x)|dx > 0$ for all $x_1 < b', x_2 > b'$.

Proof. Suppose condition (a) holds. Then (3.1.6) implies

$$(B_0'y,y) = \sum_{\nu=0}^{n} \int_0^b p_\nu(x)\,|y^{[n-\nu]}(x)|dx \ge C(y,y) \qquad (y \in D_0')$$

So the operator B', and also its closure are bounded from below by the number C, that is $(By,y) \ge C(y,y)$ $(y \in D)$, and it follows easily that the range of B is closed.

The statement is now a consequence of COROLLARY 3.1.2.

In case of condition (b), I.M. Glazman's decomposition method [11] can be used. With b' from condition (b), the restriction of the equation $y^{[2n]} = rf$ to the intervals $[0,b']$ and $[b',b)$ with boundary conditions $y^{[\nu]}(0) = y^{[\nu]}(b') = 0$, $(\nu = 0,1,\ldots, 2n-1)$ define operators $\hat{A}_1$ and $\hat{A}_2$ in $L_r^2[0,b']$ and $L_r^2[b',b)$. In other words, the operator $\hat{A}_1$ is a regular quasi-differential operator with minimal domain generated by

$$y^{[2n]} = rf \text{ on } [0,b'] \text{ and } y^{[\nu]}(0) = y^{[\nu]}(b') = 0, \quad \nu = 0,1,\ldots, 2n-1,$$

and, the operator $\hat{A}_2$ is a singular quasi-differential operator with minimal domain generated by

$$y^{[2n]} = rf, \qquad \text{and } y^{[\nu]}(b) = 0, \ \nu = 0,1,\dots, 2n-1.$$

The deficiency index of $\hat{A}_1$ ($\hat{A}_2$) is $2n$ (n respectively). Let

$$D_0'' = \{y \in D_0' : y^{[\nu]}(b') = 0, \ \nu = 0,1,\dots, 2n-1\}$$

and $\hat{A}_0'' = A'|_{D_0''}$, $\hat{A}_0$ = the closure of $\hat{A}_0''$ in $L_r^2(0,b)$. (Here $A' = JB'$, see Section 2.5.) Then $\hat{A}_0$ is the direct sum of $\hat{A}_1$ and $\hat{A}_2$ in $L_r^2(0,b)$:

$$\hat{A}_0 = \hat{A}_1 \oplus \hat{A}_2 \quad \text{in} \quad L_r^2(0,b) = L_r^2(0,b') \oplus L_r^2(b',b).$$

Moreover, $\dim(D(A)/D(\hat{A}_0)) = 2n$ i.e. the "difference" of the domains of A and $\hat{A}_0$ is $2n$-dimensional.

The J-selfadjoint extensions of the regular operator $\hat{A}_1$ have non-empty resolvent sets (see above for the second-order case). The same is true for the selfadjoint extensions of the operator $\hat{A}_2$ as $\hat{A}_2$ is symmetric with respect to a definite scalar product. Now it follows easily that e.g. condition (iii) of Lemma 3.1.1 is fulfilled.

If equation $M(y) = |r|f$ is in limit (n,n) case at b and the statements (i - iv) of Lemma 3.1.1 hold, i.e. $\rho(A_\alpha) \neq \emptyset$ for the J-selfadjoint operator A_α, $0 \leq \alpha < \pi I_n$, we shall say that <u>equation</u> $M(y) = rf$ <u>is in</u> r-<u>limit</u> (n,n) <u>case at</u> b.

3.2. Operators associated with the equivalent first order system

In Section 2.2 with the equation $M(y) = |r|f$ we associated the operator $\tilde{B}: \tilde{B}y = f$ in L_r^2. This definition makes sense under the besic assumptions I - IV, since $y = 0$ implies $f = 0$. The operator $\tilde{A}$ was defined in Section 3.1 by $\tilde{A} = J\tilde{B}$, which means that $\tilde{A}y = f$ if and only if $y^{[2n]} = rf$ for $y, f \in L^2$. In the domain $\tilde{D}$ of $\tilde{A}$ and $\tilde{B}$ we have

$$[\tilde{A}u, v] = (\tilde{B}u, v), \quad u, v \in \tilde{D}.$$

In Section 2.3 it was shown that the equation $y^{[2n]} = rf$ is equivalent to the first order system:

$$K\hat{y}'(x) = S(x)\hat{y}(x) + R(x)\hat{f}(x) . \tag{3.2.1}$$

Introducing the $(2n\times 2n)$-matrix function Z as the solution of the initial problem:

$$KZ'(x) - S(x)Z(x) = 0 \quad (0 \leq x < b) , \quad Z(0) = I_{2n} ,$$

we have (compare Section 2.7)

$$Z(x)^* KZ(x) = K \qquad (0 \leq x < b)$$

and hence

$$K^* Z^{-1}(x)K = Z(x)^* \quad (0 \leq x < b) . \tag{3.2.2}$$

Then in the same way as in Section 2.7 it follows that the general solution of (3.2.1) is

$$\hat{y}(x) = Z(x)(\hat{c} - \int_0^x Z^{-1}(t)\,KR(t)\hat{f}(t)dt) \tag{3.2.3}$$

with an arbitrary $2n$-vector $\hat{c}$.

Recall that L_R^2 is the space defined in Section 2.4 with scalar product (2.4.4) (positive definite) and (2.4.5) (indefinite). The following lemma makes it possible to associate an operator in L_R^2 with the equation (3.2.1).

LEMMA 3.2.1. Let (b_m) be the sequence from assumption IV. Then for the function $\hat{y}$ of (3.2.3) the following statement is true:
$\hat{y} = 0$ in $L_R^2[0,b_m)$ implies $\hat{f} = 0$ and $Z(x)\hat{c}$ in $L_R^2[0,b_m)$.

Observe that similar to Remark 2.2 this statement holds also with b_m replaced by b.

Proof. If $\hat{y} = 0$ in $L_R^2[0,b_m)$ then $y = 0$ in $L_r^2[0,b_m)$ (see (2.3.4)) and, by assumption IV, $f = 0$ in $L_r^2[0,b_m)$, if $y^{[2n]} = rf$. This, by (2.3.5), implies $\hat{f} = 0$ in $L_R^2[0,b_m)$ and from (3.2.3) it follows $Z(x)\hat{c} = 0$ in $L_R^2[0,b_m)$.

In L_R^2 an operator $\tilde{T}$ is defined as follows. Its domain $D(\tilde{T})$ contains all $\hat{y}$ having a representation (3.2.3) with $\hat{f}$ in L_R^2, $\hat{c} \in \mathbb{C}^{2n}$ and such that the first component y of $\hat{y}$ belongs to $D(\tilde{A})$. The operator is then defined by

$$\tilde{T}\hat{y} = \hat{f}.$$

The preceding Lemma 3.2.1 shows that $\hat{y} = 0$ in L_R^2 implies $\hat{f} = 0$ in L_R^2 so that the definition makes sense.

If for $\hat{\varphi}$ in L_R^2 we put [1]

$$\begin{aligned} &\hat{\varphi}^{(0)}(x) = \hat{\varphi}(x) \\ &\hat{\varphi}^{(j+1)}(x) = Z(x) \int_0^x Z^{-1}(t)\, KR(t)\, \hat{\varphi}^{(j)}(t)dt, \quad j = 0,1,\ldots, N-1 \end{aligned} \tag{3.2.4}$$

(here the upper index means "integration", N arbitrary integer)

the following lemma holds true.

LEMMA 3.2.2 (cf. [28]). Let N be an arbitrary natural number N. Then the set of all $\hat{f}$:

$$\begin{aligned} \hat{f}(x) = \hat{\varphi}^{(N)}(x) &= Z(x) \int_0^x Z^{-1}(t_{N-1})\, KR(t_{N-1})\, \hat{\varphi}^{(N-1)}(t_{N-1})dt_{N-1} = \\ &= Z(x) \int_0^x Z^{-1}(t_{N-1})\, KR(t_{N-1})\, Z(t_{N-1}) \int_0^{t_{N-1}} Z^{-1}(t_{N-2})\, KR(t_{N-2})dt_{N-2} \cdots \\ &\cdots Z(t_1) \int_0^{t_1} Z^{-1}(t)\, KR(t)\, \hat{\varphi}(t)dt. \end{aligned} \tag{3.2.5}$$

where $\hat{\varphi} = \hat{\varphi}^{(0)}$ belongs to L_R^2 and satisfies

$$\hat{\varphi}^{(j)}(x) = 0, \quad x \geq b', \quad b' \in (0,b), \quad j = 0,1,\ldots, N \tag{3.2.6}$$

is dense in L_R^2.

Proof. If

$$\int_0^b \langle R(x)\hat{f}(x), \hat{g}(x)\rangle dx = 0 \tag{3.2.7}$$

for all $\hat{f}$ as given in the lemma and some $\hat{g} \in L_R^2$, then it will be shown that $\hat{g} = 0$ in L_R^2.

---

1) There should be no confusion between the vector-function $\hat{\varphi}$, $\hat{\theta}$ of this section and those of Section 2.6-7.

Consider the system (3.2.4) with $\hat{f} \in L^2_R$ having representation (3.2.5) and $\hat{\varphi}$ satisfying (3.2.6) for a fixed b' $(0 < b' < b)$. Then by partial integration and using (3.2.2) one gets

$$
\begin{aligned}
0 &= \int_0^{b'} \langle R(x)\hat{f}(x)\,,\ \hat{g}(x)\rangle\,dx \\
&= \int_0^{b'} \Big\langle R(x)Z(x)\int_0^x Z^{-1}(t)\,K\,R(t)\,\hat{\varphi}^{(N-1)}(t)dt,\ \hat{g}(x)\Big\rangle\,dx \\
&= \int_0^{b'} \Big\langle \int_0^x Z^{-1}(t)\,K\,R(t)\,\hat{\varphi}^{(N-1)}(t)dt,\ Z^*(x)R(x)\hat{g}(x)\Big\rangle\,dx \\
&= \Big[\Big\langle \int_0^x Z^{-1}(t)\,K\,R(t)\,\hat{\varphi}^{(N-1)}(t)dt,\ \int_0^x Z^*(t)R(t)\hat{g}(t)dt\Big\rangle\Big]_0^{b'} - \\
&\quad - \int_0^{b'} \Big\langle Z^{-1}(x)\,K\,R(x)\,\hat{\varphi}^{(N-1)}(x),\ \int_0^x Z^*(t)R(t)\hat{g}(t)dt\Big\rangle\,dx = \\
&= \int_0^{b'} \Big\langle R(x)\,\hat{\varphi}^{(N-1)}(x),\ K(Z^{-1}(x))^*\int_0^x Z^*(t)R(t)\hat{g}(t)dt\Big\rangle\,dx \\
&= \int_0^{b'} \Big\langle R(x)\,\hat{\varphi}^{(N-1)}(x),\ K(Z^{-1}(x))^*K^*\int_0^x Z^{-1}(t)\,K\,R(t)\hat{g}(t)dt\Big\rangle\,dx \\
&= \int_0^{b'} \Big\langle R(x)\,\hat{\varphi}^{(N-1)}(x),\ K^*(Z^{-1}(x))^*K\int_0^x Z^{-1}(t)\,K\,R(t)\hat{g}(t)dt\Big\rangle\,dx \\
&= \int_0^{b'} \Big\langle R(x)\,\hat{\varphi}^{(N-1)}(x),\ Z(x)\int_0^x Z^{-1}(t)\,K\,R(t)\hat{g}(t)dt\Big\rangle\,dx \\
&= \int_0^{b'} \langle R(x)\,\hat{\varphi}^{(N-1)}(x),\ \hat{g}^{(1)}(x)\rangle\,dx = \ldots = \\
&= \int_0^{b'} \langle R(x)\,\hat{\varphi}(x),\ \hat{g}^{(N)}(x)\rangle\,dx\,. \qquad (3.2.8)
\end{aligned}
$$

Therefore $(\hat{g}^{(N)}\chi_{[0,b']})(\cdot)$ is orthogonal (with respect to the indefinite scalar product in L^2_R) to the set of all $\hat{\varphi}$ with property (3.2.6), $\chi_{[0,b']}$ is the characteristic function of the interval $[0,b']$.

According to (3.2.6) one gets for $j = 0,1,\ldots, N-1$

$$0 = \hat{\varphi}^{(j+1)}(b') = \langle \hat{\varphi}^{(j+1)}(b'), \hat{v}_j \rangle$$

$$= \langle Z(b') \int_0^{b'} Z(x)^{-1} K R(x) \hat{\varphi}^{(j)}(x)dx, \hat{v}_j \rangle$$

$$= \int_0^{b'} \langle Z(x)^{-1} K R(x) \hat{\varphi}^{(j)}(x), Z(b')^* \hat{v}_j \rangle dx$$

$$= \int_0^{b'} \langle R(x) \hat{\varphi}^{(j)}(x), Z(x) K Z(b')^* \hat{v}_j \rangle dx$$

$$= \int_0^{b'} \langle R(x) \hat{\varphi}^{(j)}(x), \psi_j(x) \rangle dx, \qquad (3.2.9)$$

where $\hat{\psi}_j(x) = Z(x) K Z(b')^* \hat{v}_j$, $\hat{v}_j$ arbitrary element in $\mathbb{C}^{2n}$. Here the relation (3.2.2) has been used. The relation (3.2.9) can be written as:

$$0 = \int_0^{b'} \langle R(x)Z(x) \int_0^x Z(t)^{-1} K R(t) \hat{\varphi}^{(j-1)}(t)dt, \hat{\psi}_j(x) \rangle dx$$

$$= \int_0^{b'} \langle R(x) \hat{\varphi}^{(j-1)}(x), Z(x) \int_0^x Z(t)^{-1} K R(t) \hat{\psi}_j(t)dt \rangle dx. \qquad (3.2.10)$$

With $\hat{\psi}_j^{(0)}(x) = \hat{\psi}_j(x)$

$$\hat{\psi}_j^{(j+1)}(x) = Z(x) \int_0^x Z(t)^{-1} K R(t) \hat{\psi}_j^{(j)}(t)dt, \quad j = 0,1,\ldots,N-1,$$

then (3.2.9) can be written as

$$0 = \int_0^{b'} \langle R(x) \hat{\varphi}(x), \hat{\psi}_j^{(j)}(x) \rangle dx. \qquad (3.2.11)$$

As (3.2.9) holds for $j = 0,1,\ldots,N-1$, we have (3.2.11) for $j = 0,1,\ldots,N-1$. Now (3.2.8) and (3.2.11) imply that $\hat{g}^{(N)}\chi_{[0,b']}$ is a linear combination of $\hat{\psi}_j^{(j)}$, which can be written as:

$$(\hat{g}^{(N)}\chi_{[0,b']})(x) - \sum_{j=0}^{N-1} \hat{\psi}_j^{(j)}(x) = 0 \quad \text{in} \quad L_R^2[0,b'). \qquad (3.2.12)$$

Here $\chi_{[0,b']}$ is the characteristic function of the interval $[0,b']$.

Now, we show that relation (3.2.12) implies $g = 0$ in L_R^2 (in the whole interval $[0,b)$). To this end, in accordance with Lemma 3.2.1, we choose $b' = b_m$ and consider first the case $N = 1$. Then we have

$$\hat{g}^{(1)} - \hat{\psi}_0^{(0)} = 0 \quad \text{in} \quad L_R^2[0,b_m)$$

which can be written as

$$Z(x)\Big(\int_0^x Z(t)^{-1}KR(t)\hat{g}(t)dt - KZ(b_m)^*\hat{v}_0\Big) = 0 \quad \text{in} \quad L_R^2[0,b_m),$$

that is $\hat{g}=0$ in $L_R^2[0,b_m)$ and $KZ(b_m)^*v_j = 0$ in $L_R^2[0,b_m)$.

In case $N=2$ we obtain

$$\hat{g}^{(2)} - \hat{\psi}_0^{(0)} - \hat{\psi}_1^{(1)} = 0 \quad \text{in} \quad L_R^2[0,b_m),$$

that is

$$Z(x)\Big(\int_0^x Z(t)^{-1}KR(t)(\hat{g}^{(1)}(t) - \hat{\psi}_1^{(0)}(t)dt - KZ(b_m)^*\hat{v}_0) = 0 \quad \text{in} \quad L_R^2[0,b_m),$$

and again by Lemma 3.2.1 we have

$$\hat{g}^{(1)} - \hat{\psi}_1^{(0)} = 0 \quad \text{in} \quad L_R^2[0,b_m).$$

This relation can be written

$$Z(x)\Big(\int_0^x Z(t)^{-1}KR(t)\hat{g}(t)dt - KZ(b_m)^*\hat{v}_1\Big) = 0$$

which gives again by Lemma 3.2.1

$$\hat{g}=0 \quad \text{in} \quad L_R^2[0,b_m) \quad \text{and} \quad KZ(b_m)^*\hat{v}_1 = 0 \quad \text{in} \quad L_R^2[0,b_m).$$

The same reasoning, applied to (3.2.12) with arbitrary N, shows that (3.2.12) holds also for $N-1,\dots,1$, and the statement follows.

COROLLARY 3.2.3. <u>The domain of an arbitrary power of the operator</u> $\tilde{T}$ <u>is dense in</u> L_R^2.

In the sequel, α is again a real and symmetric $(n\times n)$-matrix with property $0 \le \alpha < \pi I_n$, and L_R' is the space introduced in Section 2.3.

If $\hat{y} \in D(\tilde{T})$ and $\hat{y}$ vanishes at b, then from (3.2.3) we have

$$\hat{y}(b) = Z(b)\Big(\hat{c} - \int_0^b Z(t)^{-1}KR(t)\hat{f}(t)dt\Big) = 0$$

in a neighbourhood of b. This gives

$$\hat{c} = \int_0^b Z(t)^{-1}KR(t)\hat{f}(t)dt \quad \text{with some} \quad f \in L_R', \tag{3.1.13}$$

and the relation (3.2.3) can be written as

$$\hat{y}(x) = Z(x) \int_x^b Z(t)^{-1} KR(t)\hat{f}(t)dt \,, \text{ with } \hat{f} \in L'_R \,. \qquad (3.2.14)$$

Introduce the set $\hat{D}'_\alpha$ as follows:

$\hat{y} \in \hat{D}'_\alpha$ if $\hat{y}$ has a representation (3.2.14) and satisfies the initial condition at zero:

$$(\cos\alpha)\,\hat{y}_1(0) - (\sin\alpha)\,\tilde{I}_n\,\hat{y}_2(0) = 0 \,. \qquad (3.2.15)$$

We define an operator T_α as the restriction of $\tilde{T}$ to the set $\hat{D}'_\alpha$, that is:

$T_\alpha\hat{y} = \hat{f}$ means that $\hat{y}$ has representation (3.2.14) and satisfies (3.2.15).

LEMMA 3.2.4 (cf. H. Langer [28]). Let N be an arbitrary natural number. Then, for arbitrary $\hat{u} \in D(T_\alpha^N)$, there exists a sequence $(\hat{v}_n) \subset D(T_\alpha^{N+1})$ such that:

$$T_\alpha^N \hat{v}_n \to T_\alpha^N \hat{u} \text{ and } \hat{v}_n \to \hat{u} \quad (n \to \infty) \,. \qquad (3.2.16)$$

The elements $\hat{v}_n$ can be chosen such that their supports are all contained in some compact subinterval of $[0,b)$.

Proof. Choose $\hat{\theta}$, $\hat{\varphi} \in L'_R$ and set (cf. Lemma 3.5)

$$\hat{\varphi}^{\{0\}}(x) = \hat{\varphi}(x)$$

$$\hat{\varphi}^{\{j+1\}}(x) = Z(x) \int_x^b Z^{-1}(t)\, K R(t)\, \hat{\varphi}^{\{j\}}(t)dt\,, \quad j = 0,1,\ldots,N-1$$

and

$$\hat{\theta}^{\{0\}}(x) = \hat{\theta}(x)$$

$$\hat{\theta}^{\{j+1\}}(x) = Z(x) \int_x^b Z^{-1}(t)\, K R(t) \hat{\theta}^{\{j\}}(t)dt\,, \quad j = 0,1,\ldots,N\,.$$

(Here the upper index means again "integration".)

We put $\hat{u}(x) = \hat{\varphi}^{\{N\}}(x)$ and $\hat{v}(x) = \hat{\theta}^{\{N+1\}}(x)$. Then $\hat{u} \in D(T_\alpha^N)$ if

$$(\cos\alpha)\, \hat{\varphi}_1^{\{j+1\}}(0) - (\sin\alpha)\, \tilde{I}_n\, \hat{\varphi}_2^{\{j+1\}}(0) = 0\,, \quad j = 0,1,\ldots,N-1 \qquad (3.2.17)$$

and $v \in D(T_\alpha^{N+1})$ if

$$(\cos\alpha)\, \hat{\theta}_1^{\{j+1\}}(0) - (\sin\alpha)\, \tilde{I}_n \hat{\theta}_2^{\{j+1\}} = 0\,, \quad j = 0,1,\ldots,N, \qquad (3.2.18)$$

and we have

$$T_\alpha^N \hat{u} = \hat{\varphi} \quad \text{and} \quad T_\alpha^N \hat{v} = \hat{\theta}^{\{1\}}$$

For a given $\hat{u} \in D(T_\alpha^N)$, choose a compact set $[0,b']$ $(b' < b)$ of $[0,b)$, such that the support of $\hat{u}$ is contained in $[0,b')$. Denote by E the set of all $\hat{\varphi}$ with support in $[0,b')$ and belonging to L'_R with property (3.2.17); and by E_1 the set of all $\hat{\theta}^{\{1\}}$ with support in $[0,b')$ and belonging to L'_R with property (3.2.18). Then the first relation in (3.2.16) will follow if one shows that E_1 is dense in E, i.e. $\overline{E}_1 \supset E$ (here $\overline{E}_1$ indicates the closure of E_1).

Consider $\hat{f} \in L'_R$, $\hat{f} \in E_1^{[\perp]}$, i.e.

$$\int_0^{b'} \langle R(x)\hat{\theta}^{\{1\}}(x),\, \hat{f}(x)\rangle\, dx = 0 \quad \text{for all} \quad \hat{\theta}^{\{1\}} \in E_1\,.$$

More explicitly, this means

$$\int_0^{b'} \langle R(x)Z(x) \int_x^{b'} Z^{-1}(t)\, K R(t)\hat{\theta}(t)dt\,,\, \hat{f}(x)\rangle\, dx = 0$$

for all $\hat{\theta} \in L_R'$ with property (3.2.18). But this is (by partial integration and the relation (3.2.2)) equivalent to

$$0 = \int_0^{b'} \Big\langle \int_x^{b'} Z^{-1}(t)\, K R(t) \hat{\theta}(t) dt\,,\; Z^*(x) R(x) \hat{f}(x) \Big\rangle\, dx$$

$$= \int_0^{b'} \Big\langle R(x)\hat{\theta}(x)\,,\; Z(x) \int_0^{x} Z^{-1}(t)\, K R(t) \hat{f}(t) dt \Big\rangle\, dx\,. \qquad (3.2.19)$$

The relation (3.2.18) can be written for $j = 0,1,\ldots,N$

$$0 = \langle (\cos\alpha)\, \hat{\theta}_1^{\{j+1\}}(0), \hat{\eta}\rangle - \langle (\sin\alpha)\, \tilde{I}_n \hat{\theta}_2^{\{j+1\}}(0), \hat{\eta}\rangle$$

$$= \langle \tilde{I}_n \hat{\theta}_1^{\{j+1\}}(0),\, \tilde{I}_n (\cos\alpha)\, \hat{\eta}\rangle - \langle \tilde{I}_n \hat{\theta}_2^{\{j+1\}}(0),\, (\sin\alpha)\, \hat{\eta}\rangle\,,$$

and, according to the property of K, the above relation can be written as

$$0 = \langle K\, \hat{\theta}^{\{j+1\}}(0),\, \hat{w}\rangle\,,$$

where $\hat{w} = ((\sin\alpha)\hat{\eta}\,,\; \tilde{I}_n(\cos\alpha)\hat{\eta})^T$, $\hat{\eta}$ arbitrary n-vector. This relation again can be written as

$$0 = \Big\langle K \int_0^{b'} z(x)^{-1}\, K R(x) \hat{\theta}^{\{j\}}(x) dx\,,\; \hat{w}\Big\rangle$$

$$= \int_0^{b'} \langle R(x)\hat{\theta}^{\{j\}}(x)\,,\; Z(x)\hat{w}\rangle\, dx\,. \qquad (3.2.20)$$

Define:

$$\hat{\psi}^{(0)}(x) = Z(x)\, \hat{w}$$

$$\hat{\psi}^{(j+1)}(x) = Z(x) \int_0^{x} Z(t)^{-1}\, K R(t) \hat{\psi}^{(j)}(t) dt\,, \quad j = 0,1,\ldots,N-1,$$

then by partial integration (3.2.20) becomes

$$0 = \int_0^{b'} \langle R(x)\hat{\theta}(x)\,,\; \hat{\psi}^{(j)}(x)\rangle\, dx\,. \qquad (3.2.21)$$

As (3.2.20) holds for $j = 0,1,\ldots,N$, we have (3.2.21) for $j = 0,1,\ldots,N$.

Hence, according to (3.2.19) $Z(x) \int_0^{x} Z^{-1}(t)\, K R(t) f(t) dt$ is orthogonal to the set of all $\hat{\theta}$, with property (3.2.21). This implies

$$Z(x) \int_0^{x} Z^{-1}(t)\, K R(t) \hat{f}(t) dt - \sum_{j=0}^{N} c_j \hat{\psi}^{(j)}(t) = 0$$

with some constants $c_0, c_1, \ldots, c_N$ or

$$Z(x)\Big(\int_0^{x} Z^{-1}(t)\, K R(t) \hat{f}(t) dt - c_0 \hat{w} - \sum_{j=1}^{N} c_j \int_0^{x} Z^{-1}(t)\, K R(t) \hat{\psi}^{(j-1)}(t) dt\Big) = 0\,.$$

Now Lemma 3.2.1 with $b_m = b'$ implies

$$\hat{f}(t) = \sum_{j=1}^{N} c_j \hat{\psi}^{(j-1)}(t) \quad \text{in} \quad L_R^2[0,b_m).$$

Hence $\hat{f}$ is also orthogonal (with respect to the indefinite scalar-product provided in L_R^2) to the set E, i.e. $\hat{f} \in E^{[\perp]}$. So it was shown that $\hat{f} \in E_1^{[\perp]}$ implies $\hat{f} \in E^{[\perp]}$, that is $E^{[\perp]} \supset E_1^{[\perp]}$ which is equivalent to $\overline{E}_1 \supset E$.

To prove the second assertion of (3.2.16), consider the sequence $(\hat{v}_n) \subset D(T_\alpha^{N+1})$ such that the support of all elements $\hat{v}_n$ are contained in some fixed compact part of $[0,b') \subset [0,b)$. It is known from the proof of the first part of (3.2.16) that

$$T_\alpha^N \hat{v}_n \longrightarrow T_\alpha^N \hat{u} \qquad (n \to \infty) \quad \text{in} \quad L_R^2[0,b) \tag{3.2.22}$$

and one has to show

$$\hat{v}_n \longrightarrow \hat{u} \qquad (n \to \infty).$$

To this end put

$$T_\alpha^N \hat{u} = \hat{\varphi} \quad \text{and} \quad T_\alpha^N \hat{v}_n = \hat{\varphi}_n.$$

Then

$$(T_\alpha^{N-1} \hat{v}_n)(x) - (T_\alpha^{(N-1)} \hat{u})(x) = Z(x) \int_x^b Z^{-1}(t)\, KR(t)(\hat{\varphi}_n(t) - \hat{\varphi}(t))dt. \tag{3.2.23}$$

The right hand side of (3.2.23) goes to zero as $n \to \infty$ by (3.2.22). By choosing the sequence $\hat{v}_n$ one has also with $b' < b$

$$(T_\alpha^{N-1} \hat{v}_n)(x) - (T_\alpha^{N-1} \hat{u})(x) = Z(x) \int_x^{b'} Z^{-1}(t)\, KR(t)(\hat{\varphi}_n(t) - \hat{\varphi}(t))dt \longrightarrow 0$$

as $n \to \infty$ in $L_R^2(0,b')$

and by Schwartz' inequality the left hand side of this last relation goes pointwise to zero as $n \to \infty$.

If this argument is repeated N times, one gets

$$\hat{v}_n(x) \longrightarrow \hat{u}(x) \qquad (n \to \infty)$$

and this ends the proof of the lemma.

3.3. Definitizability of the operator A_α. In this section, α is again a real symmetric $(n\times n)$-matrix with the property $0 \le \alpha < \pi I_n$. We enumerate its eigenvalues such that $\pi > \mu_1(\alpha) \ge \dots \ge \mu_n(\alpha) \ge 0$; here each eigenvalue is counted according to its multiplicity.

In the following theorem, by γ_α we denote the number of negative squares of the bilinear form $[A_\alpha u,v]$, $u,v \in D(A_\alpha)$.

THEOREM 3.3.1. If the integer k is given by the relation

$$\mu_k(\alpha) > \frac{\pi}{2} \ge \mu_{k+1}(\alpha),\ k=0 \text{ if } \mu_1(\alpha) \le \frac{\pi}{2},\ k=n \text{ if } \mu_n(\alpha) > \frac{\pi}{2}, \tag{3.3.1}$$

then

$$\gamma_\alpha \le k.$$

Proof. According to (3.1.1) and the Lagrangean identity one obtains

$$[A_\alpha y,y] = \sum_{\nu=0}^{n} \int_0^b p_\nu(x)\, |y^{[n-\nu]}(x)|^2 dx + \sum_{\nu=1}^{n} y^{[2n-\nu]}(0)\, \overline{y}^{[\nu-1]}(0). \tag{3.3.2}$$

The expression $D(y) = \sum_{\nu=0}^{n} \int_0^b p_\nu(x)\, |y^{[n-\nu]}(x)|^2 dx$ is nonnegative. Hence the number of negative squares of the bilinear form $[A_\alpha u,v]$ on $D(A_\alpha)$ is at most equal to the number of negative squares of the expression $\sum_{\nu=1}^{n} y^{[2n-\nu]}(0)\, \overline{y}^{[\nu-1]}(0)$. But this expression written in the notation of subsection 2.3 is

$$\sum_{\nu=1}^{n} y^{[2n-\nu]}(0)\, \overline{y}^{[\nu-1]}(0) = \langle \tilde{I}_n \hat{y}_2, \hat{y}_1 \rangle. \tag{3.3.3}$$

Let P_0 be the projector in $\mathbb{C}^n$ onto the eigenspace corresponding to the eigenvalue zero of the matrix α and P_0' be the orthogonal projector onto the range of α. Then, multiplying the initial condition at $x=0$ by P_0 from the left, we get

$$(\cos\alpha)\, P_0 \hat{y}_1(0) - (\sin\alpha)\, P_0 \tilde{I}_n \hat{y}_2(0) = 0.$$

By the definition of P_0 it follows

$$P_0 \hat{y}_1(0) = 0.$$

Now the right hand side of (3.3.3) can be written as

$$\begin{aligned} \langle \tilde{I}_n \hat{y}_2, \hat{y}_1 \rangle &= \langle \tilde{I}_n \hat{y}_2,\ (P_0 + P_0')\hat{y}_1 \rangle \\ &= \langle P_0' \tilde{I}_n \hat{y}_2,\ P_0' \hat{y}_1 \rangle. \end{aligned} \tag{3.3.4}$$

By multiplying the initial condition at zero from the left by P_0', we get

$$(\cos\alpha)\, P_0' \hat{y}_1(0) - (\sin\alpha)\, P_0' \tilde{I}_n \hat{y}_2(0) = 0,$$

and

$$(\cot\alpha)\, P_0' \hat{y}_1(0) = P_0' \tilde{I}_n \hat{y}_2(0).$$

Here $(\cot\alpha)\, P_0' = 0$ on the nullspace of P_0'. Now (3.3.4) gives

$$\langle \tilde{I}_n \hat{y}_2, \hat{y}_1 \rangle = \langle (\cot\alpha)\, P_0' \hat{y}_1,\ P_0' \hat{y}_1 \rangle .$$

Therefore, according to (3.3.2 - 3), we have

$$[A_\alpha y, y] = D(y) + \langle (\cot\alpha)\, P_0' \hat{y}_1,\ P_0' \hat{y}_1 \rangle ,$$

and the statement follows.

LEMMA 3.3.2 (see H. Langer [28, Lemma 1.4]). Suppose L with scalar product $\{\cdot,\cdot\}$ is a pré-Pontrjagin space with index κ, T is a linear operator in L with the following properties:

(i) $\{Tu, v\} = \{u, Tv\}$ $\quad (u, v \in D(T))$.

(ii) For each $u \in L$, there exists a sequence $(u_n) \subset D(T)$ such that $\{u_n - u, v\} \to 0$ for each $v \in L$ and $\{u_n, u_n\} \to \{u, u\}$ if $n \to \infty$.

Then, there exists a polynomial q of degree $\kappa' \leq \kappa$ with the properties that all its zeros belong to $\mathbb{C}_+ \cup \mathbb{R}$ and $\{q(T)u, q(T)u\} \geq 0$ for all $u \in D(T^{\kappa'})$.

The linear space $L = D(A_\alpha)$ with the (possibly degenerated) scalar product:

$$\{u, v\} = [A_\alpha u, v] \quad (u, v \in D(A_\alpha)) \tag{3.3.5}$$

is a pré-Pontrjagin space with index $\kappa \leq \gamma_\alpha$. In this space we consider the operator $T = A_\alpha|_{D(A_\alpha^2)}$. Then the condition (i) of Lemma 3.3.2 is obviously satisfied. Moreover, by Lemma 3.2.4 and the isomorphism between the spaces L_R^2 and L_r^2 and the operators T_α and A_α', for arbitrary $u \in D(A_\alpha')$ there exists a sequence $(v_n) \subset D(A_\alpha'^2)$, such that

$A'_\alpha v_n \to A'_\alpha u$ and $v_n \to u$ $(n \to \infty)$. As in [28], Lemma 2.7, it follows from Lemma 3.3.2, that there exists a polynomial q of degree $\kappa_\alpha \le \gamma_\alpha$, such that

$$\{q(T)u,\ q(T)u\} = [A_\alpha\, q(A_\alpha)u\ ,\ q(A_\alpha)u] \ge 0 \quad (u \in D(A_\alpha^{\kappa_\alpha+1}))\ . \tag{3.3.6}$$

Suppose that the equation $M(y) = rf$ is in r-limit (n,n) case at b. Then, if q is chosen such that its degree is minimal, its zeros must be eigenvalues of the operator A_α.

In order to see this, let s denote a zero of q, i.e. $q(s) = 0$. First, suppose $s \ne \bar{s}$. If $s \notin \sigma_p(A_\alpha)$, then $s \in \rho(A_\alpha)$. Let $q(\lambda) = (\lambda - s)^K\, q_1(\lambda)$, $q_1(s) \ne 0$. According to (3.3.6) we have

$$[A_\alpha\, q(A_\alpha)u,\ q(A_\alpha)u] =$$

$$= [A_\alpha(A_\alpha - s\,I)^{K'} q_1(A_\alpha)u,\ (A_\alpha - s\,I)^{K'}\, q_1(A_\alpha)u] \ge 0 \quad (u \in D(A_\alpha^{\kappa_\alpha+1}))\ ,$$

and since $s \in \rho(A_\alpha)$, this can be written as

$$[A_\alpha q_1(A_\alpha)(A_\alpha - s\,I)^{K'}u,\ q_1(A_\alpha)(A_\alpha - s\,I)^{K'}u] \ge 0 \quad (u \in D(A_\alpha^{\kappa_\alpha+1}))\ .$$

The operator $(A_\alpha - s\,I)^{K'}$ maps $D(A_\alpha^{\kappa_\alpha+1})$ onto $D(A_\alpha^{\kappa_\alpha+1-K'})$ which gives

$$[A_\alpha q_1(A_\alpha)v,\ q_1(A_\alpha)v] \ge 0 \quad (v \in D(A_\alpha^{\kappa_\alpha+1-K'}))$$

a contradiction to the minimality of the degree of q.

If $s = \bar{s} \ne 0$, let $q(\lambda) = (\lambda - s)^{K''}\, q_2(\lambda)$, $q_2(s) \ne 0$ and choose an open bounded interval Δ such that $s \in \Delta$, $0 \notin \bar{\Delta}$. Let E_α be the eigenspectral function of the operator A_α, then an arbitrary $u \in D(A_\alpha)$ can be written as

$$u = (I - E_\alpha(\Delta))\, u + E_\alpha(\Delta)\, u\ .$$

Now

$$[A_\alpha q(A_\alpha)u,\ q(A_\alpha)u] = [A_\alpha(I - E_\alpha(\Delta))q(A_\alpha)u,\ (I - E_\alpha(\Delta))q(A_\alpha)u] + \\ + [A_\alpha q(A_\alpha)E_\alpha(\Delta)u,\ E_\alpha(\Delta)q(A_\alpha)u],\ u \in D(A_\alpha^{\kappa_\alpha+1}). \tag{3.3.7}$$

The operator $T_\alpha = A_\alpha\big|_{(I-E_\alpha(\Delta))L_r^2}$ has its spectrum in the complement of the interval Δ. On $(I - E_\alpha(\Delta))L_r^2$ one has $(T_\alpha - sI)D(T_\alpha) = (I - E_\alpha(\Delta))L_r^2$ which implies

$$[T_\alpha q_1(T_\alpha)u,\ q_1(T_\alpha)u] \geq 0 \qquad (u \in D(T_\alpha^{\kappa_\alpha+1-\kappa''})).$$

That is

$$[A_\alpha q_1(A_\alpha)u,\ q_1(A_\alpha)u] \geq 0 \qquad (u \in D(T_\alpha^{\kappa_\alpha+1-\kappa''}) \cap (I - E_\alpha(\Delta))L_r^2). \tag{3.3.8}$$

If $s \notin \sigma_p(A_\alpha)$, then $(A_\alpha - sI)D(A_\alpha)$ is dense in L_r^2 which implies $(A_\alpha - sI)D(A_\alpha^n)$ is dense in L_r^2 for any natural number n. The operator $S_\alpha = A_\alpha\big|_{E_\alpha(\Delta)L_r^2}$ is bounded and $(S_\alpha - sI)D(S_\alpha) = E_\alpha(\Delta)L_r^2$.

According to (3.3.6) one obtains

$$[A_\alpha q_1(A_\alpha)(A_\alpha - sI)^{\kappa'_\alpha}u,\ q_1(A_\alpha)(A_\alpha - sI)^{\kappa'_\alpha}u] \geq 0 \qquad (u \in D(A_\alpha^{\kappa_\alpha+1}))$$

which implies

$$[S_\alpha q_1(S_\alpha)u,\ q_1(S_\alpha)u] \geq 0 \qquad (u \in E_\alpha(\Delta)L_r^2).$$

This relation together with (3.3.8) implies

$$[A_\alpha q_1(A_\alpha)u,\ q_1(A_\alpha)u] \geq 0 \qquad (u \in D(A_\alpha^{\kappa'_\alpha+1-\kappa''})), \tag{3.3.9}$$

which is again a contradiction to the minimality of the degree of q.

The case $s = 0$ is considered in the following lemma.

LEMMA 3.3.3. Let A be a bounded J-selfadjoint operator in a Krein space H with indefinite scalar product[1] $[\cdot,\cdot]$ and $\rho(A) \neq \emptyset$. Moreover, assume there exists a natural number $n \geq 1$ such that

$$[A^n u, u] \geq 0 \tag{3.3.10}$$

If n is minimal with respect to (3.3.10) and $0 \notin \sigma_p(A)$ then

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[1] Here we suppose that $J \neq I$ (see p.1.1.1).

$[Au,u] \geq 0$ *for all* u.

Proof. Let $n = 2\nu$ $(\nu = 1,2,\ldots)$. According to (3.3.10) one gets $[A^\nu u, A^\nu u] \geq 0$ for all u which implies $A^\nu u \in \mathcal{P}_+$ (see subsection 1.1). Therefore, there exists a $v \neq 0$ such that $[A^\nu u, v] = 0$ for all u. From the J-selfadjointness of the operator A, one obtains $[u, A^\nu v] = 0$ for all u, i.e. $A^\nu v = 0$, $v \neq 0$. But this means that $0 \in \sigma_p(A)$, a contradiction to the hypothesis $0 \notin \sigma_p(A)$.

For $n = 2\nu+1$ $(\nu = 1,2,\ldots)$ one gets from (3.3.10)

$$[A A^\nu u, A^\nu u] \geq 0 \quad \text{for all} \quad u. \tag{3.3.11}$$

If $0 \notin \sigma_p(A)$ then the range of $A^\nu : \mathcal{R}(A^\nu)$ is dense in H. Let $v \in \mathcal{R}(A^\nu)$ i.e. $v = A^\nu u$ for some u. Then from (3.3.11) one has

$$[Av, v] \geq 0 \quad \text{for all} \quad v \quad \text{from the dense set} \quad (A^\nu),$$

which implies

$$[Af, f] \geq 0 \quad \text{for all} \quad f.$$

These results are summarized in the following

THEOREM 3.3.4. *Suppose the equation* $M(y) = r f$ *is in* r-*limit* (n,n) *case at* b *and* k *is given by the relation* (3.3.1) *with the real symmetric* $(n \times n)$-*matrix* α. *Then there exists a polynomial* q *of degree* $\kappa_\alpha \leq k$ *with the property*

$$[A_\alpha q(A_\alpha) u, q(A_\alpha) u] \geq 0 \quad (u \in D(A_\alpha^{\kappa_\alpha + 1})). \tag{3.3.12}$$

If q *is chosen such that its degree is minimal, it is unique up to a constant factor and, every zero of the polynomial* $P = q\bar{q}$ *is an eigenvalue of* A_α. *The non-real spectrum of* A_α *consists of a finite number of pairs* $s, \bar{s}$. *Each isolated point of the spectrum of* A_α, $\sigma(A_\alpha)$ *is an eigenvalue of finite Riesz' index* $\nu(s) = \nu(\bar{s})$ *and*

$$\nu(s) = \begin{cases} \nu(s) = n(s) & \text{if} \quad s \neq \bar{s} \\ \nu(s) \leq n(s) + 1 & \text{if} \quad s = \bar{s}, \end{cases}$$

where $n(s)$ denotes the multiplicity of the point s as a zero of the polynomial P.

In the terminology introduced of Section 1.3, A_α is definitizable with definitizing polynomial $P = q\bar{q}$.

3.3.1. The second order case. In this section, under the assumptions I, II′, III and IV′, we consider now the equation $-(\frac{1}{p}y')' + qy = rf$ and suppose it is in r-limit (1,1) case at b. With α_0 as given in Lemma 2.7, and by (3.1.2) the following statements hold true:

(i) $[A_\alpha u,v] \geq 0 \quad (u \in D(A_\alpha))$ if $\alpha \in [0,\alpha_0]$.

(ii) The bilinear form $[A_\alpha u,v] \quad (u,v \in D(A_\alpha))$ has exactly one negative square if $\alpha \in (\alpha_0,\pi)$.

Theorem 3.3.4 can be more specified.

THEOREM 3.3.5. Suppose $(-\frac{1}{p}y')' + qy = rf$ is in r-limit (1,1) case at b, i.e. the operators A_α are J-selfadjoint in L^2_r and $\rho(A_\alpha) \neq 0$. Then one has

(i′) $[A_\alpha u,u] \geq 0 \quad (u \in D(A_\alpha))$ if $\alpha \in [0,\alpha_0]$.

(ii′) If $\alpha \in (\alpha_0,\pi)$, there exists an eigenvalue $s_\alpha \neq 0$ of A_α, such that

$$[A_\alpha(A_\alpha - s_\alpha I)u_1(A_\alpha - s_\alpha I)u] \geq 0 \quad (u \in D(A_\alpha^2)).$$

The conclusion $s_\alpha \neq 0$, $\alpha \in (\alpha_0,\pi)$ follows from Lemma 2.9.5.

Here the definitizable operator A_α has the following definitizing polynomial:

$$P(z) = \begin{cases} z & \text{when } \alpha \in [0,\alpha_0], \\ z(z - s_\alpha)(z - \bar{s}_\alpha) & \text{when } \alpha \in (\alpha_0,\pi). \end{cases}$$

Consequently, for each bounded real interval Δ with $0 \notin \bar{\Delta}$ and $s_\alpha \notin \bar{\Delta}$ (when $\alpha \in (\alpha_0,\pi)$) there exists a $\gamma_\Delta > 0$ such that

$$[E_\alpha(\Delta)u,u] \geq \gamma_\Delta \|E_\alpha(\Delta)u\|^2 \quad ([E_\alpha(\Delta)u,u] \leq -\gamma_\Delta \|E_\alpha(\Delta)u\|^2)$$

when $\Delta > 0$ ($\Delta < 0$ respectively).

COROLLARY 3.3.6. Suppose $-(\frac{1}{p}y')'+qy = rf$ is in limit (1,1) case at b and $\alpha \in [0,\alpha_0]$. Then the spectrum of A_α is real, 0 and ∞ being the only possible critical points of A_α.

Let $\varphi_\alpha(\cdot,\lambda)$ denote the solution of the initial problem $-(\frac{1}{p}y')'+qy = \lambda ry$, $\varphi_\alpha(0,\lambda) = \sin\alpha$, $\varphi_\alpha^{[1]}(0,\lambda) = \cos\alpha$. If $\alpha \in (\alpha_0,\pi)$, then, evidently, $\varphi_\alpha(\cdot,s_\alpha)$ is the eigenelement of A_α corresponding to the eigenvalue s_α, and with Theorem 1.4.2 we obtain

COROLLARY 3.3.7. Suppose $-(\frac{1}{p}y')'+qy = rf$ is in r-limit (1,1) case at b and $\alpha \in (\alpha_0,\pi)$. Then exactly one of the following statements holds true:

(1) The nonreal spectrum of A_α consists of one pair of algebraically (simple) eigenvalues s_α, $\bar{s}_\alpha$ with corresponding eigenelements u_α, v_α respectively having the property

$$[u_\alpha, u_\alpha] = [v_\alpha, v_\alpha] = 0, \quad [u_\alpha, v_\alpha] \neq 0.$$

(2a) $s_\alpha = \bar{s}_\alpha$ and s_α is a regular critical point of A_α, the dimension of the corresponding algebraic eigenspace G_{s_α} being one. Then the eigenelement $u_\alpha = \varphi_\alpha(\cdot,s_\alpha) \in L_r^2$ has the property $[u_\alpha, u_\alpha] > 0$ (< 0) when $s_\alpha < 0$ (> 0 respectively).

(2b) $s_\alpha = \bar{s}_\alpha$ and s_α is a regular critical point of A_α, the dimension of G_{s_α} being 2 or 3. Then the elements

$$u_\alpha = \varphi_\alpha(\cdot,s_\alpha);\ \dot{u}_\alpha = \frac{\partial}{\partial\lambda}\varphi_\alpha(\cdot,\lambda)\big|_{\lambda=s_\alpha} \quad (\text{and } \ddot{u}_\alpha = \frac{\partial^2}{\partial\lambda^2}\varphi_\alpha(\cdot,\lambda)\big|_{\lambda=s_\alpha} \text{ resp.})$$

belong to L_r^2 and form a Jordan chain of A_α to the eigenvalue s_α. One has $[u_\alpha, u_\alpha] = 0$, but the indefinite scalar product is non-degenerate on G_{s_α}.

(2c) $s_\alpha = \bar{s}_\alpha$ and s_α is a singular critical point of A_α. Then the dimension of G_{s_α} is one or two, the elements u_α (and $\dot{u}_\alpha$ respectively) belong to L_r^2 and form a Jordan chain of A_α to the eigenvalue s_α, $[u_\alpha, u_\alpha] = 0$. The indefinite scalar product $[\cdot,\cdot]$ degenerates on G_{s_α}.

In case (2a - 2c) the spectrum $\sigma(A_\alpha)$ is real.

It can be shown that there are always infinitely many values of α, $\alpha \in (\alpha_0, \pi)$, such that s_α is non-real (see Theorem 4.7 in [6]).

3.4. Generalization of Weyl's function in the indefinite case (cf. Section 2.7).

Again, in this section the index α[1] will be omitted (cf. Section 2.7). For arbitrary $\lambda \in \mathbb{C}$, consider the solutions $\varphi_k(\cdot, \lambda)$[2], $\chi_k(\cdot, \lambda)$, $k = 0,1,\ldots n-1$, of the equation

$$y^{[2n]} = \lambda r y \quad \text{on} \quad [0,b) \tag{3.4.1}$$

such that the matrix function

$$V(\cdot,\lambda) = \begin{pmatrix} \varphi_0(\cdot,\lambda) & \cdots & \varphi_{n-1}(\cdot,\lambda) & \chi_0(\cdot,\lambda) & \cdots & \chi_{n-1}(\cdot,\lambda) \\ \vdots & & & \vdots & & \\ \varphi_0^{[n-1]}(\cdot,\lambda) & \cdots & \varphi_{n-1}^{[n-1]}(\cdot,\lambda) & \chi_0^{[n-1]}(\cdot,\lambda) & \cdots & \chi_{n-1}^{[n-1]}(\cdot,\lambda) \\ \varphi_0^{[n]}(\cdot,\lambda) & \cdots & \varphi_{n-1}^{[n]}(\cdot,\lambda) & \chi_0^{[n]}(\cdot,\lambda) & \cdots & \chi_{n-1}^{[n]}(\cdot,\lambda) \\ \vdots & & & \vdots & & \\ \varphi_0^{[2n-1]}(\cdot,\lambda) & \cdots & \varphi_{n-1}^{[2n-1]}(\cdot,\lambda) & \chi_0^{[2n-1]}(\cdot,\lambda) & \cdots & \chi_{n-1}^{[2n-1]}(\cdot,\lambda) \end{pmatrix}$$

or, with evident notation,

$$V(\cdot,\lambda) = \begin{pmatrix} \varphi(\cdot,\lambda;1) & \chi(\cdot,\lambda;1) \\ \varphi(\cdot,\lambda;2) & \chi(\cdot,\lambda;2) \end{pmatrix} = \begin{pmatrix} \varphi(\cdot,\lambda) & \chi(\cdot,\lambda) \end{pmatrix} \tag{3.4.2}$$

is a solution of the equivalent first order system

$$K \frac{d}{dx} V(x,\lambda) = (S(x) + \lambda R(x)) V(x,\lambda) \quad (x \in [0,b), \text{ a.e.}) \tag{3.4.3}$$

and satisfies the following initial condition at zero:

$$V(0,\lambda) = \begin{pmatrix} \varphi(0,\lambda;1) & \chi(0,\lambda;1) \\ \varphi(0,\lambda;2) & \chi(0,\lambda;2) \end{pmatrix} = \begin{pmatrix} (\sin\alpha) & (\cos\alpha) \\ \tilde{I}_n(\cos\alpha) & -\tilde{I}_n(\sin\alpha) \end{pmatrix}. \tag{3.4.4}$$

The matrices K, $\tilde{I}_n$, $S(x)$ and $R(x)$ have been defined in Section 2.3. Of course, V and its elements also depend on the matrix α.

According to the integral equation

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[1] Here α is again a real and symmetric $(n \times n)$-matrix, $0 \leq \alpha < \pi I_n$.

[2] There should be no confusion between the functions $\varphi_k(\cdot,\lambda)$, $k=0,1,\ldots,n-1$ and those of Section 2.7.

For fixed $x \in [0,b)$, the matrix function $V(\cdot,\lambda)$ is an entire function of λ. From the integral equation

$$V(x,\lambda) - V(0,\lambda) = \int_0^x K^{-1}(S(t)+\lambda R(t))V(t,\lambda)dt$$

it follows that for each compact set $\bigwedge \subset \mathbb{C}$ and for each $\varepsilon > 0$, there exists a $\delta_{\bigwedge}(\varepsilon)$, such that $|x| < \delta_{\bigwedge}(\varepsilon)$ implies

$$\|V(x,\lambda) - V(0,\lambda)\| \leq \varepsilon \quad ^{1)} \quad \text{for all } \lambda \in \bigwedge .$$

That is one obtains the following

LEMMA 3.4.1. For each compact set $\bigwedge$ of $\mathbb{C}$ and $\varepsilon > 0$, there exists a $\delta_{\bigwedge}(\varepsilon) > 0$ such that $|x| < \delta_{\bigwedge}(\varepsilon)$ and $\lambda \in \bigwedge$ imply

$$\|\varphi(x,\lambda;1) - (\sin\alpha)\| \leq \varepsilon, \ \|\chi(x,\lambda;1) - (\cos\alpha)\| \leq \varepsilon .$$

In the sequel L_r' denotes the space introduced in Section 2.8. Define on L_r' homogeneous and additive functionals depending on α:

$$\Phi_j(f;\lambda) = \int_0^b r(x)f(x)\varphi_j(x,\lambda)dx , \quad j = 0,1,\ldots,n-1 \qquad (3.4.5)$$

and a mapping

$$f \longmapsto \Phi(f;\lambda) = (\Phi_0(f;\lambda),\ldots, \Phi_{n-1}(f;\lambda))^T . \qquad (3.4.6)$$

In the following theorem a matrix function $W(\cdot)$, which plays the same role as the function $M(\cdot)$ in Section 2.7, is introduced.

THEOREM 3.4.2. Suppose $M(y) = rf$ is in the r-limit (n,n) case at b. Then for each real and symmetric $(n\times n)$-matrix α, with property $0 \leq \alpha < \pi I_n$ in the initial condition at $x = 0$, there exists one and only one $(n\times n)$-matrix function W:

$$W(\lambda) = (w_{jk}(\lambda))_{j,k=0}^{n-1}, \quad w_{jk}(\bar{\lambda}) = \overline{w_{kj}(\lambda)}, \ 0 \leq j, k \leq n-1 \quad \text{or} \quad W(\bar{\lambda}) = W(\lambda)^*$$

piecewise meromorphic on $\mathbb{C}_+ \cup \mathbb{C}_-$, not vanishing identically on $\mathbb{C}_+$ or $\mathbb{C}_-$, and with the property that the functions

---

1) The norm of a matrix is by definition the maximum of the moduli of its elements.

$$\psi_k(x,\lambda) = \chi_k(x,\lambda) + \sum_{j=0}^{n-1} \varphi_j(x,\lambda) w_{jk}(\lambda)$$

belong to L_r^2 for $k = 0,1,\ldots,n-1$. Moreover, for $\lambda \in \rho(A_\alpha)$, $\lambda = \overline{\lambda}$, the solution g of the equation

$$(A_\alpha - \lambda I)g = f \tag{3.4.7}$$

for arbitrary $f \in L_r^2$ has the following representation:

$$g(x,\lambda) = \sum_{k=0}^{n-1} \left(\psi_k(x,\lambda) \int_0^x r(t)f(t)\varphi_k(t,\lambda)dt + \varphi_k(x,\lambda) \int_x^b r(t)f(t)\psi_k(t,\lambda)dt\right).$$

Proof. From Theorem 3.3.5 it follows that $\mathbb{C}_+ \cup \mathbb{C}_-$ with exception of at most $2\kappa_\alpha$ $(0 \le \kappa_\alpha$ $(0 \le \kappa_\alpha \le n)$ points belongs to the resolvent set $\rho(A_\alpha)$ of A_α.

Suppose $\lambda \in \rho(A_\alpha)$, $\lambda \neq \overline{\lambda}$. Then, for each $f \in L_r^2$, the equation (3.4.7) has a unique solution $g \in L_r^2$.

(i) Consider $n+1$ functions $f_0, f_1,\ldots,f_n$ which vanish on a neighbourhood of b. Assume, e.g., that for $b' \in (0,b)$, we have

$$f_k(x) = 0 \quad \text{if} \quad x \ge b', \; k = 0,1,\ldots,n. \tag{3.4.8}$$

Then the corresponding solutions $g_0, g_1,\ldots,g_n$ in L_r^2 of (3.4.7) satisfy

$$g_k^{[2n]}(x) - \lambda r(x) g_k(x) = 0 \quad \text{if} \quad x \ge b', \; k = 0,1,\ldots,n.$$

Therefore, for $x \ge b'$ we have

$$g_k(x,\lambda) = \sum_{j=0}^{n-1} (a_j(f_k;\lambda)\chi_j(x,\lambda) + b_j(f_k;\lambda)\varphi_j(x,\lambda)), \; k=0,1,\ldots,n, \tag{3.4.9}$$

where $a_j(f_k;\lambda)$, $b_j(f_k;\lambda)$, $j = 0,1,\ldots,n-1$, $k = 0,1,\ldots,n-1$ are independent of x.

The functions $g_0, g_1,\ldots,g_n$ must be linearly dependent on $[b',b)$. Otherwise, the deficiency index of A should be equal to $n+1$. More explicitly, the equation

$$(\widetilde{A} - \lambda I)y = 0$$

would have $(n+1)$-linearly independent solutions in L_r^2.
But this is impossible, since the deficiency index of A is equal to n

and $A^+ = \tilde{A}$ (see Proposition 2.5.5 and (3.1.1)). Thus $g_0, g_1, \ldots, g_n$ are linearly dependent on $[b', b)$.

(ii) If the elements $f_0, \ldots, f_{n-1}$ are linearly independent in L^2_r, the determinant of the matrix $(a_j(f_k;\lambda))_{j,k=0}^{n-1}$ cannot be zero. Indeed, otherwise there exist numbers $d_0, \ldots, d_{n-1}$ not all zero such that

$$\sum_{k=0}^{n-1} a_j(f_k;\lambda) d_k = 0, \quad j = 0,1,\ldots,n-1.$$

Consider $f = \sum_{k=0}^{n-1} d_k f_k$, then the above relation becomes

$$a_j(f;\lambda) = 0, \quad j = 0,1,\ldots,n-1. \tag{3.4.10}$$

From (3.4.9), with $f_n = f$, we have for the corresponding solution g of (3.4.7)

$$\begin{aligned} g(x,\lambda) &= \sum_{k=0}^{n-1} d_k g_k(x,\lambda) = \\ &= \sum_{k=0}^{n-1} d_k \Big(\sum_{j=0}^{n-1} (a_j(f_k;\lambda)\chi_j(x,\lambda) + b_j(f_k;\lambda)\varphi_j(x.\lambda)) = \\ &= \sum_{j=0}^{n-1} (a_j(f;\lambda)\chi_j(x,\lambda) + b_j(f;\lambda)\varphi_j(x,\lambda)) = \\ &= \sum_{j=0}^{n-1} b_j(f;\lambda)\varphi_j(x,\lambda) \qquad (x \geq b'). \end{aligned}$$

It follows that the function

$$h = \sum_{j=0}^{n-1} \varphi_j(x,\lambda) b_j(f;\lambda) \tag{3.4.11}$$

belongs to L^2_r. It satisfies also the initial condition at $x = 0$, i.e.:

$$(\cos\alpha)\,\hat{h}_1(0) - (\sin\alpha)\,\tilde{I}_n \hat{h}_2(0) = 0. \tag{3.4.12}$$

Indeed, take the quasi-derivative on both sides of (3.4.11)

$$h^{[\nu]}(x,\lambda) = \sum_{j=0}^{n-1} \varphi_j^{[\nu]}(x,\lambda) b_j(f;\lambda), \quad \nu = 0,1,\ldots,2n-1,$$

and with

$$\hat{b} = (b_0(f;\lambda), \ldots, b_{n-1}(f;\lambda))^T$$

we find

$$\hat{h}_1(0,\lambda) = \varphi(0,\lambda;1)\,\hat{b} = (\sin\alpha)\,\hat{b}\,,$$

$$\hat{h}_2(0,\lambda) = \varphi(0,\lambda;2)\,\hat{b} = (\cos\alpha)\,\hat{b}\,,$$

and (3.4.12) follows. Thus $h(x,\lambda)$ is an eigenfunction of the operator A_α. But this means that $\lambda \notin \rho(A_\alpha)$, a contradiction to the choice of λ.

On the other hand, h cannot be zero in L^2_r, as this would imply $f = 0$, a contradiction to the linear independence of the elements $f_0,\ldots,f_{n-1}$ in L^2_r.

(iii) Suppose now that the determinant of the matrix $(a_j(f_k;\lambda))_{j,k=0}^{n-1}$ is different from zero. Without loss of generality, assume $a_j(f_k;\lambda) = \delta_{jk}$, $j = 0,1,\ldots,n-1$, $k = 0,1,\ldots,n-1$. Then, according to (3.4.9), we have

$$g_k(x,\lambda) = \chi_k(x,\lambda) + \sum_{j=0}^{n-1} b_j(f_k,\lambda)\varphi_j(x,\lambda), \text{ if } x \geq b',\ k=0,1,\ldots,n-1. \tag{3.4.13}$$

For $f \in L'_r$, $f(x) = 0$ if $x \geq b'$, the solution g of (3.4.7) has the form

$$g(x,\lambda) = \sum_{k=0}^{n-1} c_k(f,\lambda)g_k(x,\lambda) =$$

$$= \sum_{k=0}^{n-1} c_k(f,\lambda)\chi_k(x,\lambda) + \sum_{k,j=0}^{n-1} c_k(f,\lambda)b_j(f_k,\lambda)\varphi_j(x,\lambda), (x \geq b'), \tag{3.4.14}$$

where $c_k(f,\lambda)$ do not depend on x. On the other hand, we have from (3.4.9) with $g_n = g$

$$g(x,\lambda) = \sum_{j=0}^{n-1} a_j(f_n,\lambda)\chi_j(x,\lambda) + b_j(f_n,\lambda)\varphi_j(x,\lambda),\ (x \geq b'). \tag{3.4.15}$$

According to the linear independence of the sets $\{\chi_j\}$ and $\{\varphi_j\}$ it follows that

$$a_j(f_n,\lambda) = c_j(f,\lambda) \quad \text{and} \quad b_j(f_n,\lambda) = \sum_{k=0}^{n-1} c_k(f,\lambda)b_j(f_k,\lambda)\,.$$

Now put

$$w_{jk}(\lambda) = b_j(f_k,\lambda)\,, \quad j = 0,1,\ldots,n-1,\ k = 0,1,\ldots,n-1\,. \tag{3.4.16}$$

According to (3.4.13) and (3.4.16) with $\psi_k(x,\lambda) = g_k(x,\lambda)$ it follows that

$$\psi_k(x,\lambda) = \chi_k(x,\lambda) + \sum_{j=0}^{n-1} \varphi_j(x,\lambda)w_{jk}(\lambda) \tag{3.4.17}$$

and $\psi_k(\cdot,\lambda)$ belongs to L^2_r for $k=0,1,\dots,n-1$.

The matrix $W(\lambda)$ is uniquely determined by the property that the functions $\psi_k(\cdot,\lambda)$ in (3.4.17) belong to L^2_r.

Indeed, if for some matrix $\widetilde{W}$ the functions

$$\psi_k(x,\lambda) = \chi_k(x,\lambda) + \sum_{j=0}^{n-1} \varphi_j(x,\lambda)\widetilde{w}_{jk}(\lambda)$$

belong to L^2_r for $k=0,1,\dots,n-1$, it follows that

$$\sum_{j=0}^{n-1} \varphi_j(\cdot,\lambda)\,(w_{jk}(\lambda) - \widetilde{w}_{jk}(\lambda))$$

belongs to L^2_r for $k=0,1,\dots,n-1$, which implies $w_{jk} = \widetilde{w}_{jk}$ by the same reasoning as in (ii).

(iv) It is easy to verify (compare with (2.8.3)), that the solution g of (3.4.7) with the functions ψ_k from (3.4.17) can be written as

$$\begin{aligned} g(x,\lambda) &= \sum_{k=0}^{n-1} \Big(\psi_k(x,\lambda) \int_0^x r(t)f(t)\varphi_k(t,\lambda)dt + \varphi_k(x,\lambda) \int_x^b \psi_k(t,\lambda)f(t)r(t)dt\Big) = \\ &= \sum_{k,j=0}^{n-1} \Phi_j(f,\lambda)w_{jk}(\lambda)\varphi_k(x,\lambda) + \chi_k(x,\lambda) \int_0^x \varphi_k(t,\lambda)f(t)r(t)dt + \\ &+ \varphi_k(x,\lambda) \int_x^b \chi_k(t,\lambda)f(t)r(t)dt\,, \end{aligned} \tag{3.4.18}$$

where $\Phi_j(f,\lambda)$ is given by (3.4.5). Now

$$\begin{aligned} \big[(A_\alpha - \lambda I)^{-1} f,h\big] &= \big[g(\cdot,\lambda),\ f(\cdot)\big] = \\ &= \sum_{k,j=0}^{n-1} \Phi_j(f,\lambda)w_{jk}(\lambda)\Phi_k(\bar{h},\lambda) + \\ &+ \sum_{k=0}^{n-1} \int_0^b r(x)\chi_k(x,\lambda) \int_0^x r(t)f(t)\varphi_k(t,\lambda)dt\cdot\bar{h}(x)dx + \\ &+ \sum_{k=0}^{n-1} \int_0^b r(x)\varphi_k(x,\lambda) \int_x^b r(t)f(t)\chi_k(t,\lambda)dt\cdot\bar{h}(x)dx\,. \end{aligned} \tag{3.4.19}$$

Next, we show that W does not vanish identically in $\mathbb{C}_+$ or $\mathbb{C}_-$. To do this, we choose an $f \in L'_r$ such that $[f,f] \neq 0$ and observe first, that (3.4.17) implies $\overline{w_{jk}(\lambda)} = w_{kj}(\bar{\lambda})$. Hence, if W vanishes on $\mathbb{C}_+$ or $\mathbb{C}_-$, it vanishes on $\mathbb{C}_+ \cup \mathbb{C}_-$. In this case (3.4.19) would imply that $\big[(A_\alpha - \lambda I)^{-1} f,f\big]$ is in $\mathbb{C}_+ \cup \mathbb{C}_-$ equal to an entire function of λ, which

is real on the real axis.

If E_α denotes the eigenspectral function, of the operator A_α, it follows that $[E_\alpha(\Delta) f,f] = 0$ if Δ is an open bounded interval, such that its boundaries are not critical points of A_α. Hence $(A_\alpha - \lambda I)^{-1} f$ can have singularities only in the finite critical points of A_α. It follows that $f = E_\alpha(\Delta) f$, if Δ is sufficiently large. This implies $[f,f] = 0$, a contradiction to the choice of f.

The statement that W is meromorphic on $\mathbb{C}_+$ and $\mathbb{C}_-$ follows from the fact that in (3.4.19) the element f can be chosen such that $\Phi_j(f,\lambda) \not\equiv 0$ $(\lambda \in \mathbb{C}_+)$.

For the rest of Section 3.4 we assume additionally to assumption III, that there exists a neighbourhood Δ of $x = 0$ such that

$$\int_\Delta \frac{dx}{|r(x)|} < \infty. \tag{3.4.20}$$

We choose a real δ-sequence (δ_ℓ) of infinitely differentiable functions δ_ℓ, which vanish outside of Δ, and define

$$f_\ell^{(\nu)}(x) = (-1)^\nu \delta_\ell^{(\nu)}(x) r(x)^{-1}, \quad \nu = 0,1,\dots,n-1, \ \ell = 1,2,\dots .$$

Then

$$\int_0^b |f_\ell^{(\nu)}(x)|^2 |r(x)| dx = \int_0^b |\delta_\ell^{(\nu)}(x)|^2 |r(x)|^{-1} dx < \infty,$$

that is $f_\ell^{(\nu)} \in L_r^2$, $\nu = 0,1,\dots,n-1$; $\ell = 1,2,\dots$. Moreover, we have

$$\lim_{\ell \to \infty} \int_0^b r(x) f_\ell^{(\nu)}(x) \begin{Bmatrix} \varphi_j(x,\lambda) \\ \chi_j(x,\lambda) \end{Bmatrix} dx = \begin{Bmatrix} \varphi_j^{(\nu)}(0,\lambda) \\ \chi_j^{(\nu)}(0,\lambda) \end{Bmatrix}, \quad j,\nu = 0,1,\dots,n-1,$$

that is, for $\ell \to \infty$ we get

$$\begin{aligned} \Phi_j(f_\ell^{(\nu)}(x);\lambda) &\to (\sin\alpha)_{\nu j}, \quad j,\nu = 0,1,\dots,n-1, \\ \Gamma_k(f_\ell^{(\nu)}(x);\lambda) &\to (\cos\alpha)_{\nu k}, \quad k,\nu = 0,1,\dots,n-1, \end{aligned} \tag{3.4.21}$$

where $\Gamma_k(f,\lambda)$ is defined on L_r' by

$$\Gamma_k(f,\lambda) = \int_0^b r(x) f(x) \chi_k(x,\lambda) dx .$$

Here we have used the fact, that the quasi-derivatives of f up to order $n-1$ coincide with the ordinary derivatives (see Section 2.1).

We introduce for $f,h \in L'_r$ and $k=0,1,\dots,n-1$

$$S_k(f,h;\lambda) = \int_0^b r(x)\chi_k(x;\lambda) \int_0^x r(t)\varphi_k(t;\lambda)f(t)dt\ \overline{h(x)}dx +$$

$$+ \int_0^b r(x)\varphi_k(x;\lambda) \int_x^b r(t)\chi_k(t;\lambda)f(t)dt\ \overline{h(x)}dx =$$

$$= \int_0^b r(x) \int_0^x r(t)(\chi_k(x;\lambda)\varphi_k(t;\lambda) - \varphi_k(x;\lambda)\chi_k(t;\lambda))f(t)dt\ \overline{h(x)}dx +$$

$$+ \Phi_k(f;\lambda)\Gamma_k(\overline{h};\lambda)\ , \qquad (3.4.22)$$

and (3.4.19) can be written as

$$[(A_\alpha - \lambda I)^{-1}f,h] = \sum_{j,k=0}^{n-1} \Phi_j(f;\lambda)w_{jk}(\lambda)\overline{\Phi_k(h;\lambda)} + \sum_{k=0}^{n-1} S_k(f,h;\lambda)\ . \qquad (3.4.23)$$

If in (3.4.22) we put $f = f_\ell^{(\nu)}$, $h = f_\ell^{(s)}$ it follows that the first term on the right hand side of (3.4.22) for $\ell \to \infty$ tends to

$$\chi_k^{(s)}(0;\lambda)\varphi_k^{(\nu)}(0;\lambda) - \varphi_k^{(s)}(0;\lambda)\chi_k^{(\nu)}(0;\lambda) =$$

$$= (\cos\alpha)_{sk}(\sin\alpha)_{\nu k} - (\sin\alpha)_{sk}(\cos\alpha)_{\nu k}\ ,$$

hence we find

$$\sum_{k=0}^{n-1} S_k(f_\ell^{(\nu)}, f_\ell^{(s)};\lambda) = \sum_{k=0}^{n-1} (\sin\alpha)_{\nu k}(\cos\alpha)_{ks}\ . \qquad (3.4.24)$$

Here we have used the fact that α and therefore also $(\cos\alpha)$ and $(\sin\alpha)$ are real symmetric matrices.

The matrix W of Theorem 3.4.2 is now denoted by W_α, as it corresponds to the initial condition given by the real symmetric $(n\times n)$-matrix α. As in Section 2.7 it can be seen, that for two such matrices α and β between $W_\alpha(\lambda)$ and $W_\beta(\lambda)$ still a relation of type (2.7.10) remains true, in particular, we have

$$W_0(\lambda) = -(W_{\frac{\pi}{2}I_n}(\lambda))^{-1}\ , \qquad (3.4.25)$$

$$W_\alpha(\lambda) = ((\cos\alpha)W_0(\lambda) + (\sin\alpha))((\cos\alpha) - (\sin\alpha)W_0(\lambda))^{-1}\ . \qquad (3.4.26)$$

With the matrix function W_α, define $\hat{W}_\alpha$ by

$$\hat{W}_\alpha(\lambda) = \begin{cases} \lambda W_\alpha(\lambda) + (\cot\alpha), & 0 < \alpha < \pi I_n, \\ \lambda^{-1} W_0(\lambda), & \alpha = 0. \end{cases} \tag{3.4.27}$$

According to (3.4.25) we obtain

$$\frac{\hat{W}_0(\lambda) - \hat{W}_0(\mu)^*}{\lambda - \bar{\mu}} = \frac{\lambda^{-1}W_0(\lambda) - (\mu^{-1}W_0(\mu))^*}{\lambda - \bar{\mu}} =$$

$$= \frac{(\mu W_{\frac{\pi}{2}I_n}(\mu))^{*-1} - (\lambda W_{\frac{\pi}{2}I_n}(\lambda))^{-1}}{\lambda - \bar{\mu}} = \tag{3.4.28}$$

$$= (\hat{W}_{\frac{\pi}{2}I_n}(\mu))^{*-1} \frac{\hat{W}_{\frac{\pi}{2}I_n}(\lambda) - \hat{W}_{\frac{\pi}{2}I_n}(\mu)^*}{\lambda - \bar{\mu}} (\hat{W}_{\frac{\pi}{2}I_n}(\lambda))^{-1} .$$

THEOREM 3.4.4. Suppose that the equation $M(y) = rf$ is in r-limit (n,n) case at b and (3.4.20) is satisfied. Then

$\hat{W}_0$ belongs to $N_0(\mathbb{C}^n)$,

and, if zero is not an eigenvalue of the real symmetric $(n\times n)$-matrix α, then

$\hat{W}_\alpha$ belongs to $N_\kappa(\mathbb{C}^n)$, where $\kappa \le \gamma_\alpha$.

Here γ_α has the same meaning as in Section 3.3.

Proof. Similarly to Lemma 2.7.1 it follows that $\hat{W}_{\frac{\pi}{2}I_n} \in N_0(\mathbb{C}^n)$ and according to the relation (3.4.28) the first part of the theorem is obtained. It remains to prove that $W_\alpha \in N_\kappa(\mathbb{C}^n)$ if all eigenvalues of the natrix α are greater than zero.

If in (3.4.23) we put $f = f_\ell^{(\nu)}$, $h = f_\ell^{(s)}$, it follows from (3.4.21) and (3.2.24) that:

$$\lim_{\ell \uparrow \infty} \left[(A_\alpha - \lambda I)^{-1} f_\ell^{(\nu)}, f_\ell^{(s)}\right] = \sum_{j,k=0}^{n-1} (\sin\alpha)_{\nu j} w_{jk}(\lambda)(\sin\alpha)_{ks} +$$

$$+ \sum_{k=0}^{n-1} (\sin\alpha)_{\nu k}(\cos\alpha)_{ks} = ((\sin\alpha)\left[W_\alpha(\lambda) + (\cot\alpha)\right](\sin\alpha))_{\nu s}$$

$$W_\alpha(\lambda) + (\cot\alpha) = \lim_{\ell \uparrow \infty} (\sin\alpha)^{-1} \left(\left[(A_\alpha - \lambda I)^{-1} f_\ell^{(\nu)}, f_\ell^{(s)}\right]\right)_{\nu,s=0}^{n-1} (\sin\alpha)^{-1}$$

Now, by (3.4.23), with $R_\lambda^\alpha = (A - \lambda I)^{-1}$ we have

$$\frac{\hat{W}_\alpha(\lambda) - (\hat{W}_\alpha(\mu))^*}{\lambda - \bar{\mu}} = \lim_{\ell \uparrow \infty} (\sin\alpha)^{-1} \left(\left[A_\alpha R_\lambda^\alpha f_\ell^{(\nu)}, R_\mu^\alpha f_\ell^{(s)} \right] \right)_{\nu,s=0}^{n-1} (\sin\alpha)^{-1}. \quad (3.4.29)$$

Let $z_1, \ldots, z_m \in \mathbb{C}^n$, $\lambda_1, \ldots, \lambda_m \in \mathbb{C}_+ \cup \mathbb{C}_-$ and $\xi_1, \ldots, \xi_m$ arbitrary complex numbers. Then, from (3.4.29) one has

$$\left\langle \frac{\hat{W}_\alpha(\lambda_i) - \hat{W}_\alpha(\bar{\lambda}_j)}{\lambda_i - \bar{\lambda}_j} z_i, z_j \right\rangle = \lim_{\ell \uparrow \infty} \left\langle (\sin\alpha)^{-1} \left(\left[A_\alpha R_{\lambda_j}^\alpha f_\ell^{(\nu)}, R_{\lambda_j}^\alpha f^{(s)} \right] \right)_{\nu,s=0}^{n-1} (\sin\alpha)^{-1} z_i, z_j \right\rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathbb{C}^n$ (see Section 2.6).

Put $(\sin\alpha)^{-1} z_i = Y_i$, $i = 1,2,\ldots,m$. Then

$$\left\langle \frac{\hat{W}_\alpha(\lambda_i) - \hat{W}_\alpha(\bar{\lambda}_j)}{\lambda_i - \bar{\lambda}_j} z_i, z_j \right\rangle = \lim_{\ell \uparrow \infty} \left\langle \left(\left[A_\alpha R_{\lambda_i}^\alpha f_\ell^{(\nu)}, R_{\lambda_i}^\alpha f_\ell^{(s)} \right] \right)_{\nu,s=0}^{n-1} Y_i, Y_j \right\rangle$$

and with $Y_i = (y_i^{(0)}, \ldots, y_i^{(n-1)})^T$, one gets

$$\left\langle \frac{\hat{W}_\alpha(\lambda_i) - \hat{W}_\alpha(\bar{\lambda}_j)}{\lambda_i - \bar{\lambda}_j} z_i, z_j \right\rangle = \lim_{\ell \uparrow \infty} \sum_{\nu,s=0}^{n-1} \left[A_\alpha R_{\lambda_i}^\alpha f_\ell^{(\nu)}, R_{\lambda_j}^\alpha f_\ell^{(s)} \right] y_i^{(\nu)} y_j^{(s)} =$$

$$= \lim_{\ell \uparrow \infty} \left[A_\alpha R_{\lambda_i}^\alpha \sum_{\nu=0}^{n-1} f_\ell^{(\nu)} y_i^{(\nu)}, R_{\lambda_j}^\alpha \sum_{s=0}^{n-1} f_\ell^{(s)} y_j^{(s)} \right].$$

Hence

$$\sum_{i,j=1}^{m} \left\langle \frac{\hat{W}_\alpha(\lambda_i) - \hat{W}_\alpha(\bar{\lambda}_j)}{\lambda_i - \bar{\lambda}_j} z_i, z_j \right\rangle \xi_i \bar{\xi}_j =$$

$$= \lim_{\ell \uparrow \infty} \left[A_\alpha \sum_{i=1}^{m} R_{\lambda_i}^\alpha \sum_{\nu=0}^{n-1} f_\ell^{(\nu)} y_i^{(\nu)} \xi_i, \sum_{j=1}^{m} R_{\lambda_j}^\alpha \sum_{s=0}^{n-1} f_\ell^{(s)} y_j^{(s)} \xi_j \right],$$

and the statement follows.

REMARK 3.4.5. From (3.4.26) and (3.4.27) it is easy to show that $\hat{W}_\alpha$ belongs to $N_{\kappa'}(\mathbb{C}^n)$ for some integer κ' which is at most equal to the number of eigenvalues of α in $(\pi/2, \pi)$. Indeed,

$$\frac{\hat{W}_\alpha(\lambda) - \hat{W}_\alpha(\mu)^*}{\lambda - \bar{\mu}} = \frac{1}{\lambda - \bar{\mu}} \left(\lambda W_\alpha(\lambda) + \lambda(\cot\alpha) - \bar{\mu} W_\alpha(\mu)^* - \bar{\mu}(\cot\alpha) \right) =$$

$$= (\cot\alpha) + (\lambda - \bar{\mu})^{-1} \left(\lambda W_\alpha(\lambda) - \bar{\mu} W_\alpha(\mu)^* \right) =$$

$$= (\cot\alpha) + (\lambda - \mu)^{-1} \{ \lambda ((\cos\alpha) W_0(\lambda) + (\sin\alpha)) ((\cos\alpha) - (\sin\alpha) W_0(\lambda))^{-1}$$

$$- \bar{\mu} ((\cos\alpha) - W_0(\mu)^*(\sin\alpha))^{-1} (W_0(\mu)^*(\cos\alpha) + (\sin\alpha)) \}.$$

Therefore,

$$\frac{\hat{W}_\alpha(\lambda) - \hat{W}_\alpha(\mu)^*}{\lambda - \bar{\mu}} = (\cot\alpha) + (\lambda - \bar{\mu})^{-1}((\cos\alpha) - W_0(\mu)^*(\sin\alpha))^{-1} \cdot$$

$$\cdot \{\lambda((\cos\alpha) - W_0(\mu)^*(\sin\alpha))((\cos\alpha) W_0(\lambda) + (\sin\alpha)) -$$

$$- \bar{\mu}(W_0(\mu)^*(\cos\alpha) + (\sin\alpha))((\cos\alpha) - (\sin\alpha) W_0(\lambda))\} \cdot ((\cos\alpha) - (\sin\alpha) W_0(\lambda))^{-1} =$$

$$= ((\cos\alpha) - W_0(\mu)^*(\sin\alpha))^{-1}\{((\cos\alpha) - W_0(\mu)^*(\sin\alpha))(\cot\alpha)((\cos\alpha) - (\sin\alpha)W_0(\lambda)) +$$

$$+ (\cos\alpha)(\sin\alpha) - W_0(\mu)^*(\sin\alpha)(\cos\alpha) W_0(\lambda) +$$

$$+ (\lambda - \bar{\mu})^{-1}\left[\lambda((\cos^2\alpha)W_0(\lambda) - W_0(\mu)^*(\sin^2\alpha)) - \bar{\mu}(W_0(\mu)^*(\cos^2\alpha) - (\sin^2\alpha)W_0(\lambda))\right]\} \cdot$$

$$\cdot ((\cos\alpha) - (\sin\alpha) W_0(\lambda))^{-1} .$$

The expression in $\{ \cdots \}$ can be written as

$$(\cot\alpha) + \frac{1}{\lambda - \bar{\mu}}(-(\cos^2\alpha)W_0(\lambda)(\lambda - \bar{\mu}) - W_0(\mu)^*(\cos^2\alpha)(\lambda - \bar{\mu}) +$$

$$+ \lambda((\cos^2\alpha)W_0(\lambda) - W_0(\mu)^*(\sin^2\alpha)) - \mu(W_0(\mu)^*(\cos^2\alpha) - (\sin^2\alpha)W_0(\lambda)))\frac{\lambda\bar{\mu}}{\lambda\bar{\mu}} =$$

$$= (\cot\alpha) + \lambda\bar{\mu}\frac{\lambda^{-1}W_0(\lambda) - \bar{\mu}^{-1}W_0(\mu)^*}{\lambda - \bar{\mu}} =$$

$$= (\cot\alpha) + \lambda\bar{\mu}\frac{\hat{W}_0(\lambda) - \hat{W}_0(\mu)^*}{\lambda - \bar{\mu}} .$$

Finally, one obtains

$$\frac{\hat{W}_\alpha(\lambda) - \hat{W}_\alpha(\mu)^*}{\lambda - \bar{\mu}} =$$

$$= ((\cos\alpha) - W_0(\mu)^*(\sin\alpha))^{-1}\{(\cot\alpha) + \lambda\bar{\mu}\frac{\hat{W}_0(\lambda) - \hat{W}_0(\mu)^*}{\lambda - \bar{\mu}}\}((\cos\alpha) - W_0(\lambda)(\sin\alpha))^{-1} .$$

By (3.4.26) the kernel

$$\frac{\hat{W}_0(\lambda) - \hat{W}_0(\mu)^*}{\lambda - \bar{\mu}}$$

is positive definite and the statement follows.

3.4.1. The second-order case. In this section we again suppose that the assumptions I, II′, III and IV′ are satisfied for the equation $M(y) = rf$. As $n = 1$ the function W_α will now be denoted by w_α. In this case, using α_0 from Lemma 2.9.4, Theorem 3.4.4 can be more specified.

THEOREM 3.4.6. *Suppose the second-order equation* $M(y) = rf$ *is in r-limit* (1,1) *case at* b. *Then*

$$\hat{w}_\alpha \in N_0(\mathbb{C}) \text{ when } \alpha \in [0,\alpha_0] \text{ and } \hat{w}_\alpha \in N_1(\mathbb{C}) \text{ when } \alpha \in (\alpha_0,\pi).$$

Proof. We first remark that in case $n = 1$, in Theorem 3.4.4, we do not need the additional condition (3.4.20), as the sequence $(f_\ell^{(0)}) = (f_\ell)$ can be chosen as

$$f_\ell(x) = \begin{cases} \operatorname{sgn} r(x) \left(\int_0^{b_\ell} |r(x)|dx \right)^{-1}, & 0 \le x \le b_\ell \\ 0, & x > b_\ell \end{cases}$$

for some sequence (b_ℓ), $b_\ell \to 0$ $(\ell \to \infty)$. Then Theorem 3.4.4 implies $\kappa \le 1$.

It remains to show that $\kappa \ge 1$ when $\alpha \in (\alpha_0,\pi)$. For arbitrary $f \in L_r'$ and by (3.4.23) it follows

$$[A_\alpha(A_\alpha - \lambda I)^{-1} f, f] = \hat{w}_\alpha(\lambda)\ p(\lambda) + q(\lambda) \tag{3.4.30}$$

with entire functions $p(\lambda) = \Phi_\alpha(f;\lambda)\ \Phi_\alpha(\bar{f};\lambda)$ and $q(\lambda)$. Here $\Phi_\alpha(f;\lambda) = \int_0^b r(x)f(x)\varphi_\alpha(x,\lambda)dx$ where $\varphi_\alpha(\cdot,\lambda)$ has the same meaning as in Section 3.3.1. Evidently, $p(\lambda) > 0$ if $\lambda = \bar{\lambda}$. The left hand side of (3.4.30) for suitable f is a function of class $N_1(\mathbb{C})$ by (ii) of Section 3.3.1 for the second-order case.

Moreover, if one chooses a compact interval Δ containing the eigenvalue s_α (see Theorem 3.3.5) in its interior when $s_\alpha = \bar{s}_\alpha$, $0 \notin \bar{\Delta} \setminus \Delta$ and with E_α as the eigenspectral function of A_α, then

$$\lambda \to [A_\alpha(A_\alpha - \lambda I)^{-1} E_\alpha(\bar{\mathbb{R}} \setminus \Delta) f,\ E_\alpha(\bar{\mathbb{R}} \setminus \Delta) f]$$

is a function of class $N_0(\mathbb{C})$, and

$$\lambda \to [A_\alpha(A_\alpha - \lambda I)^{-1}(I - E_\alpha(\bar{\mathbb{R}} \setminus \Delta))f,\ (I - E_\alpha(\bar{\mathbb{R}} \setminus \Delta))f]$$

is of class $N_1(\mathbb{C})$ holomorphic and vanishing at infinity. Therefore, in

the representation (1.6.1) of $\tilde{Q}(\lambda) = [A_\alpha(A_\alpha - \lambda I)^{-1} f, f]$ one has $\tilde{\rho}_0 = 0$, $\tilde{R}_0 = 0$ and Lemma 1.10 implies $\kappa \geq 1$.

COROLLARY 3.4.7. When $\alpha \in [0, \alpha_0]$, the function $\hat{w}_\alpha$ admits the following representation

$$\hat{w}_\alpha(\lambda) = c_0 + c_1\lambda + \int_{-\infty}^{\infty} \left(\frac{1}{t-\lambda} - \frac{t+\lambda}{1+t^2}\right)(1+t^2)\,d\sigma_\alpha(t).$$

When $\alpha \in (\alpha_0, \pi)$, the function $\hat{w}_\alpha$ admits one and only one of the following representation

(1) $s_\alpha \neq \bar{s}_\alpha$

$$\hat{w}_\alpha(\lambda) = c_0 + c_1\lambda + \int_{-\infty}^{\infty} \left(\frac{1}{t-\lambda} - \frac{t+\lambda}{1+t^2}\chi_\infty(t)\right)(1+t^2)\,d\sigma_\infty(t) + \frac{a}{\lambda - s_\alpha} + \frac{\bar{a}}{\lambda - \bar{s}_\alpha};$$

(2) $s_\alpha = \bar{s}_\alpha$

(2a) $$\hat{w}_\alpha(\lambda) = c_0 + c_1\lambda + \int_{-\infty}^{\infty} \left(\frac{1}{t-\lambda} - \frac{t+\lambda}{1+t^2}\chi_\infty(t)\right)(1+t^2)\,d\sigma_\alpha(t) + \frac{b_1}{\lambda - s_\alpha};$$

(2b) $$\hat{w}_\alpha(\lambda) = c_0 + c_1\lambda + \int_{-\infty}^{\infty} \left(\frac{1}{t-\lambda} - \frac{t+\lambda}{1+t^2}\chi_\infty(t)\right)(1+t^2)\,d\sigma_\alpha(t) + \sum_{j=1}^{3} \frac{d_j}{(s_\alpha - \lambda)^j}$$

(2c) $$\hat{w}_\alpha(\lambda) = c_0 + c_1\lambda + \int_{-\infty}^{\infty} \left(\frac{1}{t-\lambda} - \frac{t+\lambda}{1+t^2}\chi_\infty(t) + \sum_{\nu=1}^{2} \frac{(t-s_\alpha)^{\nu-1}}{(\lambda - s_\alpha)^\nu}\chi_\infty(t)\right)\frac{(1+t^2)^2}{(t-s_\alpha)^2}\,d\sigma_\alpha(t) + \sum_{j=1}^{3} \frac{d_j}{(s_\alpha - \lambda)^j}.$$

Here σ_α is a left continuous non-decreasing bounded function, continuous in s_α when $s_\alpha = \bar{s}_\alpha$ and $\int \chi_\alpha(t)(t-s_\alpha)^{-2}\,d\sigma_\alpha(t) = \infty$ in case (2c); $c_1, d_3 > 0$; $b_1 < 0$; c_0, d_1, d_2 are real, a is a complex and $|d_2| + |d_3| \neq 0$ in case (2b); χ_α (χ_∞) is the characteristic function of a neighbourhood Δ_α (Δ_∞) of s_α (∞ respectively), $\Delta_\alpha \cap \Delta_\infty = \emptyset$, $0 \in \bar{\Delta}_\alpha$.

Of course, the coefficient $a, \ldots, d_3$ also depend on α and on the choice of Δ_∞ and Δ_α.

The corollary follows from the results mentioned in Section 1.6 and Theorem 3.4.6. It only remains to show that in case $\hat{w}_\alpha \in N_1(\mathbb{C})$ either the critical point t_1 or one of the non-real poles $\beta_1, \overline{\beta}_1$ of $\hat{w}_\alpha$ (in the representation 1.6.1) coincide with s_α. Relation (3.4.19) implies that $\hat{w}_\alpha$ has a non-real pole if and only if A_α has non-real eigenvalues and these eigenvalues are $s_\alpha, \overline{s}_\alpha$. If A_α does not have a non-real eigenvalue, s_α and the spectrum of A_α is real, $s_\alpha \neq 0$, choose a bounded open interval Δ_α, $s_\alpha \in \Delta_\alpha$, $0 \notin \overline{\Delta}_\alpha$. As a function of class $N_1(\mathbb{C})$, $\hat{w}_\alpha$ has a representation (1.6.1) with $\kappa = 1$, $s = 0$. Suppose, the critical point of $\hat{w}_\alpha$ is ∞ or $t_1 \neq s_\alpha$. One may even assume $t_1 \notin \overline{\Delta}_\alpha$. Multiplication of (3.4.30) by $\frac{1}{\lambda - \overline{z}}$, $z \neq \overline{z}$, and application of Stieltjes-Livsic inversion formula (see Appendix) gives for $f \in L_r^2$ with compact support in $[0,b)$

$$[A_\alpha(A_\alpha - zI)^{-1} E_\alpha(\Delta)f,f] = \int_{\Delta_\alpha} \frac{|\Phi_\alpha(f;t)|^2 P(t)}{t - z} \, d\sigma_\alpha(t) .$$

Here P is a positive continuous function on Δ_α.

The right hand side of this relation, as a function of z belongs to the class $N_0(\mathbb{C})$. The left hand side cannot be of class $N_0(\mathbb{C})$ for all f considered, otherwise $[A_\alpha E_\alpha(\Delta_\alpha)f,f] \geq 0$ $(f \in L_r^2)$, but in the proof of Theorem 3.3.4 it was shown that this is impossible.